

The $3x + 1$ Problem: An Annotated Bibliography (1963–1999)

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ABSTRACT. The $3x + 1$ problem concerns iteration of the map $T : \mathbb{Z} \rightarrow \mathbb{Z}$ given by

$$T(x) = \begin{cases} \frac{3x+1}{2} & \text{if } x \equiv 1 \pmod{2} . \\ \frac{x}{2} & \text{if } x \equiv 0 \pmod{2} . \end{cases}$$

The $3x + 1$ Conjecture asserts that each $m \geq 1$ has some iterate $T^{(k)}(m) = 1$. This is an annotated bibliography of work done on the $3x + 1$ problem and related problems from 1963 through 1999. At present the $3x + 1$ Conjecture remains unsolved.

1. Introduction

The $3x + 1$ problem is most simply stated in terms of the *Collatz function* $C(x)$ defined on integers as “multiply by three and add one” for odd integers and “divide by two” for even integers. That is,

$$C(x) = \begin{cases} 3x+1 & \text{if } x \equiv 1 \pmod{2} , \\ \frac{x}{2} & \text{if } x \equiv 0 \pmod{2} , \end{cases}$$

The $3x + 1$ *problem*, or *Collatz problem*, is to prove that starting from any positive integer, some iterate of this function takes the value 1.

Much work on the problem is stated in terms of the $3x + 1$ *function*

$$T(x) = \begin{cases} \frac{3x+1}{2} & \text{if } x \equiv 1 \pmod{2} \\ \frac{x}{2} & \text{if } x \equiv 0 \pmod{2} . \end{cases}$$

The $3x + 1$ *Conjecture* states that every $m \geq 1$ has some iterate $T^{(k)}(m) = 1$.

The $3x + 1$ problem is generally attributed to Lothar Collatz and has reportedly circulated since at least the early 1950’s. It also goes under the names *Syracuse Problem*, *Hasse’s Algorithm*, *Kakutani’s Problem* and *Ulam’s Problem*. The first published reference to a $3x + 1$ -like function that I am aware of is Klamkin (1963), which concern’s Collatz’s original function, which is a permutation of the integers. Collatz has stated that he studied this function in the

1930's, see Lagarias (1985). The $3x + 1$ problem itself was reportedly described in an informal lecture of Collatz in 1950 at the International Math. Congress in Cambridge, Massachusetts (see Shanks (1965) and Trigg et al (1976)). The first journal appearance of the $3x + 1$ problem itself seems to be the text of a 1970 lecture of H. S. M. Coxeter, which appeared in Coxeter (1971). This was followed by Beeler et al (1972), Conway (1972), Gardner (1972), Kay (1972) and Ogilvy (1972). See Bryan Thwaites (1985) for his assertion to have formulated the problem in 1952. See Collatz (1986) for his assertions on formulating the $3x + 1$ problem prior to 1952.

The $3x + 1$ Conjecture was verified up to 2×10^{16} by a computation of Oliveira e Silva (1999). It has since been verified to at least 2×10^{18} in independent computations by Oliveira e Silva and Roosendaal. At present the $3x + 1$ conjecture remains unsolved.

Surveys on results on the $3x + 1$ problem during 1963–1999 include Lagarias (1985), Müller (1991), and the first chapter of Wirsching (1998a).

2. Terminology

We use the following definitions. The *trajectory* or *forward orbit* of an integer m is the set

$$\{m, T(m), T^{(2)}(m), \dots\}.$$

The *stopping time* $\sigma(m)$ of m is the least k such that $T^{(k)}(m) < m$, and is ∞ if no such k exists. The *total stopping time* $\sigma_{\infty}(m)$ is the least k such that m iterates to 1 under k applications of the function T i.e.

$$\sigma_{\infty}(m) := \inf \{k : T^{(k)}(m) = 1\}.$$

The *scaled total stopping time* or *gamma value* $\gamma(m)$ is

$$\gamma(m) := \frac{\sigma_{\infty}(m)}{\log m}$$

The *height* $h(m)$ the least k for which the Collatz function $C(x)$ has $C^{(k)}(m) = 1$. It is easy to show that

$$h(m) = \sigma_{\infty}(m) + d(m),$$

where $d(m)$ counts the number of iterates $T^{(k)}(m)$ in $0 \leq k < \sigma_{\infty}(m)$ that are odd integers. Finally, the function $\pi_a(x)$ counts the number of n with $|n| \leq x$ whose forward orbit under T includes a .

3. Bibliography

This annotated bibliography mainly covers research articles and survey articles on the $3x + 1$ problem and related problems. It provides additional references to earlier history, much of it appearing as problems. It also includes a few influential technical reports that were never published. Chinese references are written with surnames first.

1. Sergio Albeverio, Danilo Merlini and Remiglio Tartini (1989), *Una breve introduzione a diffusioni su insiemi frattali e ad alcuni esempi di sistemi dinamici semplici*, Note di matematica e fisica, Edizioni Cerfim Locarno **3** (1989), 1–39.

This paper discusses dimensions of some simple fractals, starting with the Sierpinski gasket and the Koch snowflake. These arise as a fixed set from combining several linear iterations. It then considers the Collatz function iteration as analogous, as it is given by set of two linear transformations. It looks at the tree of inverse iterates ("chalice") and estimates empirically the number of leaves at depth at most k as growing like c^k for $c \approx 1.265$. It discusses various cascades of points that arrive at the cycle $\{1, 4, 2\}$.

It then looks at iteration schemes of type: start with vertices of equilateral triangle, and a new point x_0 not on the triangle. Then form $f(x_0) = \frac{x_0 + x_k}{\xi}$ where $\xi > 0$ is a parameter. then iterate with k varying in any order through $1, 2, 3$. Depending on the value of ξ for $1 < \xi < 2$ get a fractal set, for $\xi = 2$ get an orbit supported on Sierpinski gasket. It discusses critical exponents, analogy with Julia sets, and then the Ising model.

2. Jean-Paul Allouche (1979), *Sur la conjecture de "Syracuse-Kakutani-Collatz"*, Séminaire de Théorie des Nombres 1978–1979, Exposé No. 9, 15pp., CNRS Talence (France), 1979. (MR 81g:10014).

This paper studies generalized $3x + 1$ functions of the form proposed by Hasse. These have the form

$$T(n) = T_{m,d,R}(n) := \begin{cases} \frac{mn + r_j}{d} & \text{if } n \equiv j \pmod{d}, 1 \leq j \leq d-1 \\ \frac{x}{d} & \text{if } n \equiv 0 \pmod{d} . \end{cases}$$

in which the parameters (d, m) satisfy $d \geq 2$, $\gcd(m, d) = 1$, and the set $R = \{r_j : 1 \leq j \leq d-1\}$ has each $r_j \equiv -mj \pmod{d}$. The author notes that it is easy to show that all maps in Hasse's class with $1 \leq m < d$ have a finite number of cycles, and for these maps all orbits eventually enter one of these cycles. Thus we may assume that $m > d$.

The paper proves two theorems. Theorem 1 improves on the results of Heppner (1978). Let $T(\cdot)$ be a function in Hasse's class with parameters d, m . Let $a > 1$ be fixed and set $k = \lfloor \frac{\log x}{a \log m} \rfloor$. Let A, B be two rationals with $A < B$ that are not of the form $\frac{m^i}{d^j}$ for integers $i \geq 0, j \geq 1$, and consider the counting function

$$F_{a,A,B}(x) := \sum_{n=1}^x \chi_{(A,B)} \left(\frac{T^{(k)}(n)}{n} \right),$$

in which $\chi_{(A,B)}(u) = 1$ if $A < u < B$ and 0 otherwise. Let C be the maximum of the denominators of A and B . Then:

- (i) If $A < B < 0$ then

$$F_{a,A,B}(x) = O\left(Cx^{\frac{1}{a}}\right),$$

where the constant implied by the O -symbol is absolute.

- (ii) If $B > 0$ and there exists $\epsilon > 0$ such that

$$\frac{d-1}{d} \geq \frac{\log d}{\log m} + \frac{\log B}{k \log m} + \epsilon,$$

then for $k > k_0(\epsilon)$, there holds

$$F_{a,A,B}(x) = O\left(Cx^{\frac{1}{a}} + x^{1 - \frac{|\log \eta|}{a \log m}}\right)$$

for some η with $0 < \eta < 1$ which depends on ϵ . This is true in particular when $m > d^{\frac{d}{d-1}}$ with B fixed and $x \rightarrow \infty$.

(iii) If $B > A > 0$ and if there exists ϵ such that

$$\frac{d-1}{d} \leq \frac{\log d}{\log m} + \frac{\log A}{k \log m} - \epsilon,$$

then for $k > k_0(\epsilon)$, there holds

$$F_{a,A,B}(x) = O\left(Cx^{\frac{1}{a}} + x^{1-\frac{|\log \eta|}{a \log m}}\right)$$

for some η with $0 < \eta < 1$ which depends on ϵ . This is true in particular when $m < d^{\frac{d}{d-1}}$, with A fixed and $x \rightarrow \infty$.

Theorem 2 constructs for given values d, m with $\gcd(m, d) = 1$ with $m > d \geq 2$ two functions $F(\cdot)$ and $G(\cdot)$ which fall slightly outside Hasse's class and have the following properties:

- (i) The function $F(\cdot)$ has a finite number of periodic orbits, and every n when iterated under $F(\cdot)$ eventually enters one of these orbits.
- (ii) Each orbit of the function $G(\cdot)$ is divergent, i.e. $|G^{(k)}(n)| \rightarrow \infty$ as $k \rightarrow \infty$, for all but a finite number of initial values n .

To define the first function $F(\cdot)$ pick an integer u such that $d < m < d^u$, and set

$$F(x) = \begin{cases} \frac{mx + r_j}{d} & \text{if } n \equiv j \pmod{d^u}, \gcd(j, d) = 1, \\ \frac{x}{d} + s_j & \text{if } x \equiv j \pmod{d^u}, j \equiv 0 \pmod{d}. \end{cases}$$

in which $0 < r_j < d^u$ is determined by $mj + r_j \equiv 0 \pmod{d^u}$, and $0 \leq s_j < d^{u-1}$ is determined by $\frac{j}{d} + r_j \equiv 0 \pmod{d^{u-1}}$. The second function $G(x)$ is defined by $G(x) = F(x) + 1$.

These functions fall outside Hasse's class because each is linear on residue classes $n \pmod{d^u}$ for some $u \geq 2$, rather than linear on residue classes \pmod{d} . However both these functions exhibit behavior qualitatively like functions in Hasse's class: There is a constant C such that

$$|F(n) - \frac{n}{d}| \leq C \quad \text{if } n \equiv 0 \pmod{d}.$$

$$|F(n) - \frac{mn}{d}| \leq C \quad \text{if } n \not\equiv 0 \pmod{d}.$$

and similarly for $G(n)$, taking $C = d^{u-1} + 1$. The important difference is that functions in Hasse's class are mixing on residue classes $\pmod{d^k}$ for all powers of k , while the functions $F(\cdot)$ and $G(\cdot)$ are not mixing in this fashion. The nature of the non-mixing behaviors of these functions underlies the proofs of properties (i), resp. (ii) for $F(\cdot)$, resp. $G(\cdot)$.

3. Amal S. Amleh, Edward A. Grove, Candace M. Kent, and Gerasimos Ladas (1998), *On some difference equations with eventually periodic solutions*, J. Math. Anal. Appl. **223** (1998), 196–215. (MR 99f:39002)

The authors study the boundedness and periodicity of solutions of the set of difference equations

$$x_{n+1} = \begin{cases} \frac{1}{2}(\alpha x_n + \beta x_{n-1}) & \text{if } x \equiv 0 \pmod{2} . \\ \gamma x_n + \delta x_{n-1} & \text{if } x \equiv 1 \pmod{2} , \end{cases}$$

where the parameters $\alpha, \beta, \gamma, \delta \in \{-1, 1\}$, and the initial conditions (x_0, x_1) are integers. There are 16 possible such iterations. Earlier Clark and Lewis (1995) considered the case $(\alpha, \beta, \gamma, \delta) = (1, 1, 1, -1)$, and showed that all orbits with initial conditions integers (x_0, x_1) with $\gcd(x_0, x_1) = 1$ converge to one of three periodic orbits. Here the authors consider all 16 cases, showing first a duality between solutions of $(\alpha, \beta, \gamma, \delta)$ and $(-\alpha, \beta, -\gamma, \delta)$, taking a solution $\{x_n\}$ of one to $\{(-1)^{n+1}x_n\}$ of the other. This reduces to considering the eight cases with $\alpha = +1$. They resolve six of these cases, as follows, leaving open the cases $(\alpha, \beta, \gamma, \delta) = (1, -1, 1, 1)$ and $(1, -1, -1, -1)$.

For the parameters $(1, 1, 1, 1)$ they show all orbits are eventually constant or unbounded, and that unbounded orbits occur.

For the parameters $(1, 1, 1, -1)$, the work of Clark and Lewis (1995) showed all orbits are eventually periodic.

For the parameters $(1, 1, -1, 1)$ all solutions are eventually periodic, and there are five relatively prime cycles, the fixed points $(1), (-1)$, the 4-cycles $(2, -1, 3, 1), (-2, 1, -3, 1)$ and the 6-cycle $(1, 0, 1, -1, 0, -1)$.

For the parameters $(1, 1, -1, -1)$ all solutions are eventually periodic, and there are four relatively prime cycles, the fixpoints $(1), (-1)$ and the 3-cycles $(-1, 0, 1), (1, 0, -1)$.

For parameters $(1, -1, 1, -1)$, all orbits are eventually periodic, with two relatively prime cycles, the 6-cycle $(-1, 0, 1, 1, 0, -1)$.

For parameters $(1, -1, -1, 1)$ all orbits are eventually periodic, with one relatively prime cycle, the 8-cycle $(0, -1, 1, 1, 0, 1, -1, -1)$.

The authors conjecture that for the two remaining cases $(1, -1, 1, 1)$ and $(1, -1, -1, -1)$, that all orbits are eventually periodic.

4. S. Anderson (1987), *Struggling with the $3x+1$ problem*, Math. Gazette **71** (1987), 271–274.

This paper studies simple analogues of the $3x + 1$ function such as

$$g(x) = \begin{cases} x + k & \text{if } x \equiv 1 \pmod{2} , \\ \frac{x}{2} & \text{if } x \equiv 0 \pmod{2} . \end{cases}$$

For $k = 1$ when iterated this map gives the binary expansion of x . The paper also

reformulates the $3x + 1$ Conjecture using the function:

$$f(x) = \begin{cases} \frac{x}{3} & \text{if } x \equiv 0 \pmod{3}, \\ \frac{x}{2} & \text{if } x \equiv 2 \text{ or } 4 \pmod{6}, \\ 3x + 1 & \text{if } x \equiv 1 \pmod{6}. \end{cases}$$

5. Stefan Andrei and Cristian Masalagiu (1998), *About the Collatz Conjecture*, Acta Informatica **35** (1998), 167–179. (MR 99d:68097).

This paper describes two recursive algorithms for computing $3x + 1$ -trees, starting from a given base node. A $3x + 1$ -tree is a tree of inverse iterates of the function $T(\cdot)$. The second algorithm shows a speedup of a factor of about three over the “naive” first algorithm.

6. David Applegate and Jeffrey C. Lagarias (1995a), *Density Bounds for the $3x + 1$ Problem I. Tree-Search Method*, Math. Comp., **64** (1995), 411–426. (MR 95c:11024)

Let $n_k(a)$ count the number of integers n having $T^{(k)}(n) = a$. Then for any $a \not\equiv 0 \pmod{3}$ and sufficiently large k , $(1.299)^k \leq n_k(a) \leq (1.361)^k$. Let $\pi_k(a)$ count the number of $|n| \leq x$ which eventually reach a under iteration by T . If $a \not\equiv 0 \pmod{3}$ then $\pi_a(x) > x^{.643}$ for all sufficiently large x . The extremal distribution of number of leaves in $3x + 1$ trees with root a and depth k (under iteration of T^{-1}) as a varies are computed for $k \leq 30$. The proofs are computer-intensive.

7. David Applegate and Jeffrey C. Lagarias (1995b), *Density Bounds for the $3x + 1$ Problem II. Krasikov Inequalities*, Math. Comp., **64** (1995). 427–438. (MR 95c:11025)

Let $\pi_a(x)$ count the number of $|n| \leq x$ which eventually reach a under iteration by T . If $a \not\equiv 0 \pmod{3}$, then $\pi_a(x) > x^{.809}$ for all sufficiently large x . It is shown that the inequalities of Krasikov (1989) can be used to construct nonlinear programming problems which yield lower bounds for the exponent γ in $\pi_a(x) > x^\gamma$. The exponent above was derived by computer for such a nonlinear program having about 20000 variables.

8. David Applegate and Jeffrey C. Lagarias (1995c), *On the distribution of $3x + 1$ trees*, Experimental Mathematics **4** (1995), 101–117. (MR 97e:11033).

The extremal distribution of the number of leaves in $3x + 1$ trees with root a and depth k (under iteration of T^{-1}) as a varies were computed for $k \leq 30$ in Applegate and Lagarias (1995a). These data appear to have a much narrower spread around the mean value $(\frac{4}{3})^k$ of leaves in a $3x + 1$ tree of depth k than is predicted by (repeated draws from) the branching process models of Lagarias and Weiss (1992). Rigorous asymptotic results are given for the branching process models.

The paper also derives formulas for the expected number of leaves in a $3x + 1$ tree of depth k whose root node is $a \pmod{3^\ell}$. A 3-adic limit is proved to exist almost everywhere as $k \rightarrow \infty$, the expected number of leaves being $W_\infty(a) \left(\frac{4}{3}\right)^k$ where the function

$W_\infty : \mathbb{Z}_3^\times \rightarrow \mathbb{R}$ almost everywhere satisfies the 3-adic functional equation

$$W_\infty(\alpha) = \frac{3}{4} \left(W_\infty(2\alpha) + \psi(\alpha \bmod 9) W_\infty\left(\frac{2\alpha - 1}{3}\right) \right), \quad (*)$$

in which $\psi(\alpha) = 1$ if $\alpha \equiv 2$ or $8 \pmod{9}$ and is 0 otherwise. (Here $\mathbb{Z}_3^* = \{\alpha \in \mathbb{Z}_3 : \alpha \not\equiv 0 \pmod{3}\}$). It is conjectured that W_∞ is continuous and everywhere nonzero. It is an open problem to characterize solutions of the functional equation (*).

9. Jacques Arsac (1986), *Algorithmes pour vérifier la conjecture de Syracuse*, C. R. Acad. Sci. Paris **303**, Serie I, no. 4, (1986), 155–159. [Also: RAIRO, Inf. Théor. Appl. **21** (1987), 3–9.] (MR 87m:11128).

This paper studies the computational complexity of algorithms to compute stopping times of $3x + 1$ function on all integers below a given bound x .

10. Charles Ashbacher (1992), *Further Investigations of the Wondrous Numbers*, J. Recreational Math. **24** (1992), 1–15.

This paper numerically studies the “MU” functions $F_D(x)$ of Wiggin (1988) on $x \in \mathbb{Z}^+$ for $2 \leq D \leq 12$. It finds no exceptions for Wiggin’s conjecture that all cycles of F_D on \mathbb{Z}^+ contain an integer smaller than D , for $x < 1.4 \times 10^7$. It tabulates integers in this range that have a large stopping time, and observes various patterns. These are easily explained by observing that, for $n \not\equiv 0 \pmod{D}$, $F_D^{(2)}(n) = \frac{n(D+1)-R}{D}$ if $n \equiv R \pmod{D}$, $-1 \leq R \leq D - 2$, hence, for most n , $F_D^{(2)}(n) > n$, although F_D decreases iterates on the average.

11. Arthur Oliver Lonsdale Atkin (1966), *Comment on Problem 63 – 13**, SIAM Review **8** (1966), 234–236.

This comment gives more information of the problem of Klamkin (1963) concerning iteration of the original Collatz function, which is a permutation of the positive integers. Adding to the comment of Shanks (1965), he notes there is a method which in principle can determine all the cycles of a given period p of this map. This method determines upper and lower bounds on the integers that can appear in such a cycle. By computer calculation he shows that aside from the known cycles of periods 1, 2, 5 and 12 on the nonnegative integers, there are no other cycles of period less than 200.

He gives an example casting some doubt on the heuristic of Shanks (1965) concerning the possible lengths of periods. He shows that for the related permutation $f(3n) = 4n + 3$, $f(3n + 1) = 2n$, $f(3n + 2) = 4n + 1$, which should obey a similar heuristic, that it has a cycle of period 94 (least term $n = 140$), and 94 is not a denominator of the continued fraction convergent to $\log_2 3$.

Atkin presents a heuristic argument asserting that the Collatz permutation should only have a finite number of cycles, since the iterates grow “on average” at an exponential rate.

12. Michael R. Avidon (1997), *On primitive 3-smooth partitions of n* , Electronic J. Combinatorics **4** (1997), no.1, 10pp. (MR 98a:11136).

The author studies the number $r(n)$ of representations of n as sums of numbers of the form $2^a 3^b$ which are primitive (no summand divides another). Iterates of $3x + 1$ function applied to n that get to 1 produces a representation of n of this kind. The author proves results about the maximal and average order of this function. See also Blecksmith, McCallum and Selfredge (1998) for more information.

13. Claudio Baiocchi (1998), *3N+1, UTM e Tag-Systems* (Italian), Dipartimento di Matematica dell'Università "La Sapienza" di Roma **98/38** (1998).

This technical report constructs small state Turing machines that simulate the $3x + 1$ problem. Let $T(k, l)$ denote the class of one-tape Turing machines with k state, with l -symbols, with one read head, and the tape is infinite in two directions. The author constructs Turing machines for simulating the $3x + 1$ iteration in the classes $T(10, 2), T(5, 3), T(4, 4), T(3, 5)$ and $T(2, 8)$, working on unary inputs. It follows that no method is currently known to decide the reachability problem for such machines. The author then produces a universal Turing machine in the class $T(22, 2)$.

Note. This work was motivated by a conference paper of M. Margenstern (1998), whose journal version is: M. Margenstern, *Theor. Comp. Sci.* **231** (2000), 217-251.

14. Ranan B. Banerji (1996), *Some Properties of the $3n + 1$ Function*, *Cybernetics and Systems* **27** (1996), 473-486.

The paper derives elementary results on forward iterates of the $3x + 1$ function viewed as binary integers, and on backward iterates of the map g taking odd integers to odd integers, given by

$$g(n) := \frac{3n+1}{2^k}, \text{ where } 2^k \parallel 3n+1.$$

Integers $n \equiv 0 \pmod{3}$ have no preimages under g . If $n \not\equiv 0 \pmod{3}$ define $g^{-1}(n)$ to be the unique integer t such that $g(t) = n$ and $t \not\equiv 5 \pmod{8}$. Note that each odd n there are infinitely many \tilde{t} with $g(\tilde{t}) = n$. If $d(n) = 4n + 1$, these preimages are just $\{d^{(j)}g^{-1}(n) : j \geq 1\}$. The ternary expansion of $g^{-1}(n)$ is asserted to be computable from the ternary expansion of n by a finite automaton. The author conjectures that given any odd integer n , there is some finite k such that $(g^{-1})^{(k)}(n) \equiv 0 \pmod{3}$. Here we are iterating the partially defined map

$$\begin{aligned} g^{-1}(6n+1) &= 8n+1, \\ g^{-1}(6n+5) &= 4n+3, \end{aligned}$$

and asking if some iterate is $0 \pmod{3}$. The problem resembles Mahler's Z -number iteration [J. Australian Math. Soc. **8** (1968), 313-321].

15. Enzo Barone (1999), *A heuristic probabilistic argument for the Collatz sequence*. (Italian), *Ital. J. Pure Appl. Math.* **4** (1999), 151-153. (MR 2000d:11033).

This paper presents a heuristic probabilistic argument which argues that iterates of the $3x + 1$ -function should decrease on average by a multiplicative factor $(\frac{3}{4})^{1/2}$ at each step. Similar arguments appear earlier in Lagarias (1985) and many other places, and trace back to the original work of Terras, Everett and Crandall.

16. Michael Beeler, William Gosper and Richard Schroepfel (1972), *HAKMEM*, Memo 239, Artificial Intelligence Laboratory, MIT, 1972. (ONR Contract N00014-70-A-0362-0002).

The influential memorandum, never published in a journal, is a collection of problems and results. The list contains solved and unsolved problems, computer programs to write, programming hacks, computer hardware to design. There are 191 items in all. Example: "**Problem 95:** Solve *chess*. There are about 10^{40} possible positions; in most of them, one side is hopelessly lost."

It contains one of the earliest statements of the $3x+1$ problem, which appears as item 133. It was contributed by A. I. Lab members Richard Schroepfel, William Gosper, William Henneman and Roger Banks. It asks if there are any other cycles on the integers other than the five known ones. It asks if any orbit diverges.

[This memo is currently available online at:

<http://www.inwap.com/pdp10/hbaker/hakmen/hakmem.html>]

17. Edward Belaga (1998), *Reflecting on the $3x+1$ Mystery: Outline of a Scenario- Improbable or Realistic?* U. Strasbourg report 1998-49, 10 pages.
(<http://hal.archives-ouvertes.fr/IRMA-ACF>, file hal-0012576)

This is an expository paper, discussing the possibility that the $3x+1$ conjecture is an undecidable problem. Various known results, pro and con, are presented.

18. Edward Belaga and Maurice Mignotte (1999), *Embedding the $3x+1$ Conjecture in a $3x+d$ Context*, Experimental Math. **7**, No. 2 (1999), 145–151. (MR 200d:11034).

The paper studies iteration on the positive integers of the $3x+d$ function

$$T(x) = \begin{cases} \frac{3x+d}{2} & \text{if } x \equiv 1 \pmod{2}, \\ \frac{x}{2} & \text{if } x \equiv 0 \pmod{2}, \end{cases}$$

where $d \geq -1$ and $d \equiv \pm 1 \pmod{6}$. It proves that there is an absolute constant c such that there are at most dk^c periodic orbits which contain at most k odd integers. Furthermore c is effectively computable. This follows using a transcendence result of Baker and Wüstholz [J. reine Angew. **442** (1993), 19–62.]

19. Stefano Beltraminelli, Danilo Merlini and Luca Rusconi (1994), *Orbite inverse nel problema del $3n+1$* , Note di matematica e fisica, Edizioni Cerfim Locarno **7** (1994), 325–357.

This paper discusses the tree of inverse iterates of the Collatz function, which it terms the "chalice." It states the Collatz conjecture, and notes that it fails for the map

$$C_{3,5}(x) := \begin{cases} 3x+5 & \text{if } x \equiv 1 \pmod{2}, \\ \frac{x}{2} & \text{if } x \equiv 0 \pmod{2}, \end{cases}$$

where there are at least two periodic orbits $\{1, 8, 4, 2\}$ and $\{5, 20, 10, 5\}$; here orbits with initial term $x \equiv 0 \pmod{5}$ retain this property throughout the iteration. It studies patterns of inverse iterates in the tree with numbers written in binary.

20. Lothar Berg and Günter Meinardus (1994), *Functional equations connected with the Collatz problem*, Results in Math. **25** (1994), 1–12. (MR 95d:11025).

The $3x + 1$ Conjecture is stated as Conjecture 1. The paper proves its equivalence to each of Conjectures 2 and 3 below, which involve generating functions encoding iterations of the $3x + 1$ function $T(x)$. For $m, n \geq 0$ define $f_m(z) = \sum_{n=0}^{\infty} T^{(m)}(n)z^n$ and $g_n(w) = \sum_{m=0}^{\infty} T^{(m)}(n)w^m$. The paper shows that each $f_m(z)$ is a rational function of form

$$f_m(z) = \frac{p_m(z)}{(1 - z^{2^m})^2},$$

where $p_m(z)$ is a polynomial of degree $2^{m+1} - 1$ with integer coefficients. Conjecture 2 asserts that each $g_n(w)$ is a rational function of the form

$$g_n(w) = \frac{q_n(w)}{1 - w^2},$$

where $q_n(w)$ is a polynomial with integer coefficients, with no bound assumed on its degree. Concerning functional equations, the authors show first that the $f_m(z)$ satisfy the recursions

$$f_{m+1}(z^3) = f_m(z^6) + \frac{1}{3z} \sum_{j=0}^2 \omega^j f_m(\omega^j z^2),$$

in which $\omega := \exp\left(\frac{2\pi i}{3}\right)$ is a nontrivial cube root of unity. They also consider the bivariate generating function $F(z, w) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} T^{(m)}(n)z^n w^m$, which converges for $|z| < 1$ and $|w| < \frac{2}{3}$ to an analytic function of two complex variables. The authors show that it satisfies the functional equation

$$F(z^3, w) = \frac{z^3}{(1 - z^3)^2} + wF(z^6, w) + \frac{w}{3z} \sum_{j=0}^2 \omega^j F(\omega^j z^2, w).$$

They prove that this functional equation determines $F(z, w)$ uniquely, i.e. there is only one analytic function of two variables in a neighborhood of $(z, w) = (0, 0)$ satisfying it. Next they consider a one-variable functional equation obtained from this one by formally setting $w = 1$ (note this falls outside the known region of analyticity of the function), and dropping the non-homogenous term. This functional equation is

$$h(z^3) = h(z^6) + \frac{1}{3z} \sum_{j=0}^2 \omega^j h(\omega^j z^2).$$

Conjecture 3 asserts that the only solutions $h(z)$ of this functional equation that are analytic in the unit disk $|z| < 1$ are $h(z) = c_0 + c_1\left(\frac{z}{1-z}\right)$ for complex constants c_0, c_1 .

21. Lothar Berg and Günter Meinardus (1995), *The $3n+1$ Collatz Problem and Functional Equations*, Rostock Math. Kolloq. **48** (1995), 11–18. (MR 97e:11034).

This paper reviews the results of Berg and Meinardus (1994) and adds some new results. The first new result considers the functional equation

$$h(z^3) = h(z^6) + \frac{1}{3z} \sum_{j=0}^2 \omega^j h(\omega^j z^2).$$

with $\omega := \exp\left(\frac{2\pi i}{3}\right)$, and shows that the only solutions $h(z)$ that are entire functions are constants. The authors next transform this functional equation to an equivalent system of two functional equations:

$$\begin{aligned} h(z) + h(-z) &= 2h(z^2) \\ h(z^3) - h(-z^3) &= \frac{2}{3z} \sum_{j=0}^2 \omega^j h(\omega^j z^2). \end{aligned}$$

They observe that analytic solutions on the open unit disk to these two functional equations can be studied separately. The second one is the most interesting. Set $\Phi(z) := \int_0^z h(z) dz$ for $|z| < 1$. Making the change of variable $z = e^{\frac{2\pi i}{3}\xi}$, the unit disk $|z| < 1$ is mapped to the upper half plane $\text{Im}(\xi) > 0$. Letting $\phi(\xi) := \Phi(e^{\frac{2\pi i}{3}\xi})$ the second functional equation above becomes

$$\phi(3\xi) + \phi(3\xi + \frac{3}{2}) = \phi(2\xi) + \phi(2\xi + 1) + \phi(2\xi + 2).$$

Here we also require $\phi(\xi) = \phi(\xi + 3)$. The authors remark that it might also be interesting to study solutions to this functional equation for ξ on the real axis. The paper concludes with new formulas for the rational functions $f_m(z)$ studied in Berg and Meinardus (1994).

22. Daniel J. Bernstein (1994), *A Non-Iterative 2-adic Statement of the $3x + 1$ Conjecture*, Proc. Amer. Math. Soc., **121** (1994), 405–408. (MR 94h:11108).

Let \mathbb{Z}_2 denote the 2-adic integers, and for $x \in \mathbb{Z}_2$ write $x = \sum_{i=0}^{\infty} 2^{d_i}$ with $0 \leq d_0 < d_1 < d_2 < \dots$. Set $\Phi(x) = -\sum_{j=0}^{\infty} \frac{1}{3^{j+1}} 2^{d_j}$. The map Φ is shown to be a homeomorphism of the 2-adic integers to itself, which is the inverse of the map Q_{∞} defined in Lagarias (1985). The author proves in Theorem 1 a result equivalent to $\Phi^{-1} \circ T \circ \Phi = S$, where T is the $(3x + 1)$ -function on \mathbb{Z}_2 , and S is the shift map

$$S(x) = \begin{cases} \frac{x-1}{2} & \text{if } x \equiv 1 \pmod{2}, \\ \frac{x}{2} & \text{if } x \equiv 0 \pmod{2}. \end{cases}$$

He shows that the $3x + 1$ Conjecture is equivalent to the conjecture that $\mathbb{Z}^+ \subseteq \Phi(\frac{1}{3}\mathbb{Z})$. He rederives the known results that $\mathbb{Q} \cap \mathbb{Z}_2 \subseteq Q_{\infty}(\mathbb{Q} \cap \mathbb{Z}_2)$, and that Q_{∞} is nowhere differentiable, cf. Müller (1991).

23. Daniel J. Bernstein and Jeffrey C. Lagarias (1996), *The $3x + 1$ Conjugacy Map*, Canadian J. Math., **48** (1996), 1154–1169. (MR 98a:11027).

This paper studies the map $\Phi : \mathbb{Z}_2 \rightarrow \mathbb{Z}_2$ of Bernstein (1994) that conjugates the 2-adic shift map to the $3x + 1$ function. This is the inverse of the map Q_{∞} in Lagarias (1985); see also Akin (2004). The map $\bar{\Phi}_n \equiv \Phi \pmod{2^n}$ is a permutation of $\mathbb{Z}/2^n\mathbb{Z}$. This permutation is shown to have order 2^{n-4} for $n \geq 6$. Let $\hat{\Phi}_n$ denote the restriction of this permutation to $(\mathbb{Z}/2^n\mathbb{Z})^* = \{x : x \equiv 1 \pmod{2}\}$. The function Φ has two odd fixed points $x = -1$ and $x = 1/3$ and the 2-cycle $\{1, -1/3\}$, hence each $\hat{\Phi}_n$ inherits two 1-cycles and a 2-cycle coming from these points. Empirical evidence indicates that $\hat{\Phi}_n$ has about

$2n$ fixed points for $n \leq 1000$. A heuristic argument based on this data suggests that -1 and $1/3$ are the only odd fixed points of Φ . The analogous conjugacy map $\Phi_{25,-3}$ for the ‘ $25x - 3$ ’ problem is shown to have no nonzero fixed points.

24. Jacek Błażewicz and Alberto Pettorossi (1983), *Some properties of binary sequences useful for proving Collatz’s conjecture*, J. Found. Control Engr. **8** (1983), 53–63. (MR 85e:11010, Zbl. 547.10000).

This paper studies the $3x + 1$ Problem interpreted as a strong termination property of a term rewriting system. They view the problem as transforming binary strings into new binary strings and look in particular at its action on the patterns 1^n , 0^n and $(10)^n$ occurring inside strings. The $3x + 1$ map exhibits regular behavior relating these patterns.

25. Richard Blecksmith, Michael McCallum, and John L. Selfridge (1998), *3-Smooth Representations of Integers*, American Math. Monthly **105** (1998), 529–543. (MR 2000a:11019).

A *3-smooth representation* of an integer n is a representation as a sum of distinct positive integers each of which has the form $2^a 3^b$, and no term divides any other term. This paper proves a conjecture of Erdos and Lewin that for each integer t all sufficiently large integers have a 3-smooth representation with all individual terms larger than t . They note a connection of 3-smooth representations to the $3x + 1$ -problem, which is that a number m iterates to 1 under the $3x + 1$ function if and only if there are positive integers e and f such that $n = 2^e - 3^f m$ is a positive integer that has a 3-smooth representation with f terms in which there is one term exactly divisible by each power of three from 0 to $f - 1$. The choice of e and f is not unique, if it exists.

26. Corrado Böhm and Giovanna Sontacchi (1978), *On the existence of cycles of given length in integer sequences like $x_{n+1} = x_n/2$ if x_n even, and $x_{n+1} = 3x_n + 1$ otherwise*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. **64** (1978), 260–264. (MR 83h:10030)

The authors are primarily concerned with cycles of a generalization of the $3x + 1$ function. They consider the recursion in which $x_{n+1} = ax_n + b$, if a given recursive predicate $P(x_n)$ is true, and $x_{n+1} = cx_n + d$ if the predicate $P(x_n)$ false, where a, b, c, d and x_n are rational numbers. They observe that as a consequence of linearity alone there are at most 2^k possible cycles of period k , corresponding to all possible sequences of “true” and “false” of length n . Furthermore one can effectively determine the 2^n rationals that are the solutions to each of these equations and check if they give cycles. Thus in principle one can determine all cycles below any given finite bound. They observe that a rational number x in a cycle of the $3x + 1$ -function $T(\cdot)$ of period n necessarily has the form

$$x = \frac{\sum_{k=0}^{n-1} 3^{m-k-1} 2^{v_k}}{2^n - 3^m}$$

with $0 \leq v_0 < v_1 < \dots < v_m = n$. They deduce that every integer x in a cycle of length n necessarily has $|x| < 3^n$.

Further study of rational cycles of the $3x + 1$ function appears in Lagarias (1990).

27. David W. Boyd (1985), *Which rationals are ratios of Pisot sequences?*, Canad. Math. Bull. **28** (1985), 343–349. (MR 86j:11078).

The Pisot sequence $E(a_0, a_1)$ is defined by $a_{n+2} = \left\lfloor \frac{a_{n+1}^2}{a_n} + \frac{1}{2} \right\rfloor$, where a_0, a_1 are integer starting values. If $0 < a_0 < a_1$ then $\frac{a_n}{a_{n+1}}$ converges to a limit θ as $n \rightarrow \infty$. The paper asks: which rationals $\frac{p}{q}$ can occur as a limit? If $\frac{p}{q} > \frac{q}{2}$ then $\frac{p}{q}$ must be an integer. If $\frac{p}{q} < \frac{q}{2}$ the question is related to a stopping time problem resembling the $3x + 1$ problem.

28. Monique Bringer (1969), *Sur un problème de R. Queneau*, Mathématiques et Sciences Humaines [Mathematics and Social Science], **27** Autumn 1969, 13–20.

This paper considers a problem proposed by Queneau (1963) in connection with rhyming patterns in poetry. It concerns, for a fixed $n \geq 2$, iteration of the map on the integers

$$\delta_n(x) := \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ \frac{2n+1-x}{2} & \text{if } x \text{ is odd} \end{cases}$$

This map acts as a permutation on the integers $\{1, 2, \dots, n\}$ and it also has the fixed point $\delta_n(0) = 0$. It is called by the author a spiral permutation of $\{1, 2, \dots, n\}$. The paper studies for which n this spiral is a cyclic permutation, and calls such numbers *admissible*.

The motivation for this problem was that this permutation for $n = 6$ represents a poetic stanza pattern, the sestina, used in poems by an 11th century Troubadour, Arnaut Daniel. This pattern for general n was studied by Raymond Queneau (1963), who determined small values of n giving a cyclic permutation. His colleague Jacques Roubaud (1969) termed these rhyme schemes *n*-ines or *quenines*. Later he called these numbers *Queneau numbers*, cf. Robaud (1993).

In this paper the author, a student of Roubaud, shows that a necessary condition for a number n to be admissible is that $p = 2n + 1$ be prime. She shows that a sufficient condition to be admissible is that 2 be a primitive root (mod p). She deduces that if n and $2n + 1$ are both primes then n is admissible, and that the numbers $n = 2^k$ and $n = 2^k - 1$ are never admissible. Finally she shows that all $p \equiv 1 \pmod{8}$ are not admissible.

Note. The function $\delta_n(x)$ is defined on the integers, and is of $3x + 1$ type (i.e. it is a periodically linear function). Its long term behavior under iteration is analyzable because the function $\delta_n(x)$ decreases absolute value on each iteration for any x with $|x| > 2n + 1$. Thus all orbits eventually enter $-2n - 1 \leq x \leq 2n + 1$ and become eventually periodic. One can further show that all orbits eventually enter the region $\{0, 1, \dots, n\}$ on which $\delta_n(x)$ is a permutation.

29. Stefano Brocco (1995), *A Note on Mignosi's Generalization of the $3x + 1$ Problem*, J. Number Theory, **52** (1995), 173–178. (MR 96d:11025).

F. Mignosi (1995) studied the function $T_\beta : \mathbb{N} \rightarrow \mathbb{N}$ defined by

$$T_\beta(n) = \begin{cases} \lceil \beta n \rceil & \text{if } n \equiv 1 \pmod{2} \\ \frac{n}{2} & \text{if } n \equiv 0 \pmod{2} \end{cases},$$

where $\lceil x \rceil$ denotes the smallest integer $n \geq x$. He also formulated Conjecture C_β asserting that T_β has finitely many cycles and that every $n \in \mathbb{N}$ eventually enters a cycle under T_β . This paper shows that Conjecture C_β is false whenever β is a Pisot number or a Salem number. The result applies further to functions

$$T_{\beta,\alpha}(n) = \begin{cases} \lceil \beta n + \alpha \rceil & \text{if } n \equiv 1 \pmod{2}, \\ \frac{n}{2} & \text{if } n \equiv 0 \pmod{2}, \end{cases}$$

for certain ranges of values of β and α .

30. Thomas Brox (2000), *Collatz cycles with few descents*, Acta Arithmetica **92** (2000), 181–188. (MR 2001a:11032)

This paper considers the variant of the $3x + 1$ map, call it T_1 , that divides out all powers of 2 at each step. A cycle is written $\{x_1, x_2, \dots, x_m\}$ where each x_i is an odd integer. A *descent* means $|T_1(x)| < |x|$. Let $|C|$ denote the number of (odd) elements in a $3x + 1$ cycle C , and let $d(C)$ denote the number of descents in C . The author proves that the number of $3x + 1$ cycles C that have $d(C) < 2\log|C|$ is finite. In particular the number of Collatz cycles with $d(C) < r$ is finite for any fixed r . The proof uses a bound of Baker and Feldman and is in principle effective. Thus for each r there exists an algorithm to determine all cycles with $d(C) < r$. This greatly strengthens the result of R. Steiner (1977) who determined all cycles with one descent.

31. Serge Burckel (1994), *Functional equations associated with congruential functions*, Theoretical Computer Science **123** (1994), 397–406. (MR 94m:11147).

The author proves undecidability results for periodically linear functions generalizing those of Conway (1972). A periodically linear function $f : \mathbb{Z} \rightarrow \mathbb{Z}$ is one which is a linear function on each congruence class $(\text{mod } L)$ for some finite L . The author shows it is undecidable whether a given function has $f^{(k)}(1) = 0$ for some $k \geq 1$, and also whether a given function has the property: for each $n \geq 1$, some $f^{(k)}(n) = 0$. He also shows that the $3x + 1$ conjecture is equivalent to a certain functional equation having only the trivial solution over the set of all power-series $R(z) = \sum_{n=0}^{\infty} a_n z^n$ with all $a_i = 0$ or 1. The functional equation is

$$3z^3 R(z^3) - 3z^9 R(z^6) - R(z^2) - R(\omega z^2) - R(\omega^2 z^2) = 0$$

where $\omega = \exp(\frac{2\pi i}{3})$.

32. Robert N. Buttsworth and Keith R. Matthews (1990), *On some Markov matrices arising from the generalized Collatz mapping*, Acta Arithmetica **55** (1990), 43–57. (MR 92a:11016).

This paper studies maps $T(x) = \frac{m_i x - r_i}{d}$ for $x \equiv i \pmod{d}$, where $r_i \equiv i m_i \pmod{d}$. In the case where $\text{g.c.d.}(m_0, \dots, m_{d-1}, d) = 1$ it gives information about the structure of T -ergodic sets $(\text{mod } m)$ as m varies. A set $S \subseteq \mathbb{Z}$ is T -ergodic $(\text{mod } m)$ if it is a union of k congruence classes $(\text{mod } m)$, $S = C_1 \cup \dots \cup C_k$, such that $T(S) \subseteq S$ and there is an

n such that $C_j \cap T^{(n)}(C_i) \neq \emptyset$ holds for all i and j . It characterizes them in many cases. As an example, for

$$T(x) = \begin{cases} \frac{x}{2} & \text{if } x \equiv 0 \pmod{2}, \\ \frac{5x-3}{2} & \text{if } x \equiv 1 \pmod{2}, \end{cases}$$

the ergodic set $(\text{mod } m)$ is unique and is $\{n : n \in \mathbb{Z} \text{ and } (n, m, 15) = 1\}$, i.e. it is one of $\mathbb{Z}, \mathbb{Z} - 3, \mathbb{Z} - 5\mathbb{Z}$ or $\mathbb{Z} - 3\mathbb{Z} - 5\mathbb{Z}$ as m varies. An example is given having infinitely many different ergodic sets $(\text{mod } m)$ as m varies.

33. Charles C. Cadogan (1984), *A note on the $3x + 1$ problem*, Caribbean J. Math. **3** No. 2 (1984), 69–72. (MR 87a:11013).

Using an observation of S. Znam, this paper shows that to prove the $3x + 1$ Conjecture it suffices to check it for all $n \equiv 1 \pmod{4}$. This result complements the obvious fact that to prove the $3x + 1$ Conjecture it suffices to check it for all $n \equiv 3 \pmod{4}$. Korec and Znam (1987) obtained other results in this spirit, for odd moduli.

34. Charles C. Cadogan (1991), *Some observations on the $3x + 1$ problem*, Proc. Sixth Caribbean Conference on Combinatorics & Computing, University of the West Indies: St. Augustine Trinidad (C. C. Cadogan, Ed.) Jan. 1991, 84–91.

Cadogan (1984) reduced the $3x + 1$ problem to the study of its iterations on numbers $A_1 = \{n : n \equiv 1 \pmod{4}\}$. Here the author notes in particular the subclass $A = \{1, 5, 21, 85, \dots\}$ where $x_{i+1} = 1 + 4x_i$ of A_1 . He considers the successive odd numbers occurring in the iteration. He forms a two-dimensional table partitioning all odd integers in which A_1 is the first row and the first column of the table is called the anchor set (see Cadogan (1996) for more details). He observes that Cadogan (1984) showed the iteration on higher rows successively moves down rows to the first row, but from row A_1 is flung back to higher rows, except for the subclass A , which remains on the first row. He comments that after each revisit to row A_1 "the path may become increasingly unpredictable." He concludes, concerning further work: "The target set A_1 is being vigorously investigated."

35. Charles C. Cadogan (1996), *Exploring the $3x + 1$ problem I*, Caribbean J. Math. Comput. Sci. **6** (1996), 10–18. (MR 2001k:11032)

This paper studies iteration of the Collatz function $C(x)$, which is here denoted $f(x)$, and it includes most of the results of Cadogan (1991). The author gives various criteria under which trajectories will coalesce. He partitions the odd integers $2\mathbb{N} + 1 = \cup_{k=1}^{\infty} R_k$, in which $R_m = \{n \equiv 2^m - 1 \pmod{2^{m+1}}\}$. It enumerates their elements $R_{m,j} = (2^m - 1) + (j - 1)2^{m+1}$. and views these in an infinite two-dimensional array in which the j -th column C_j consists of the numbers $\{R_{m,j} : m \geq 1\}$. Cadogan (1984) showed that if $x \in R_m$ for some $m \geq 2$, then $f^{(2)}(x) \in R_{m-1}$, thus after $2m$ iterations one reaches an element of R_1 . Lemma 3.3 here observes that consecutive elements in columns are related by $R_{m+1,j} = 1 + 2R_{m,j}$. For the first set $R_1 = \{n \equiv 1 \pmod{4}\}$, Theorem 4.1 observes that if $y = 4x + 1$ then $f^{(3)}(y) = f(x)$. The author creates chains $\{x_n : n \geq 1\}$ related by $x_{n+1} = 4x_n + 1$ and calls these S -related elements. Theorem 4.2 then observes that the trajectories of S -related elements coalesce.

36. Marc Chamberland (1996), *A Continuous Extension of the $3x + 1$ Problem to the Real Line*, Dynamics of Continuous, Discrete and Impulsive Dynamical Systems, **2** (1996), 495–509. (MR 97f:39031).

This paper studies the iterates on \mathbb{R} of the function

$$\begin{aligned} f(x) &= \frac{\pi x}{2} \left(\cos \frac{\pi x}{2} \right)^2 + \frac{3x+1}{2} \left(\sin \frac{\pi x}{2} \right)^2 \\ &= x + \frac{1}{4} - \left(\frac{x}{2} + \frac{1}{4} \right) \cos \pi x . \end{aligned}$$

which interpolates the $3x + 1$ function $T(\cdot)$. A fact crucial to the analysis is that f has negative Schwartzian derivative $Sf = \frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2$ on \mathbb{R}^+ . On the interval $[0, \mu_1)$, where $\mu_1 = 0.27773\dots$ all iterates of f contract to a fixed point 0. Here μ_n denotes the n -th positive fixed point of f . The interval $[\mu_1, \mu_3]$ is invariant under f , where $\mu_3 = 2.44570\dots$ and this interval includes the trivial cycle $A_1 = \{1, 2\}$. On this interval almost every point is attracted to one of two attracting cycles, which are A_1 and $A_2 = \{1.19253\dots, 2.13865\dots\}$. There is also an uncountable set of measure 0 on which the dynamics is “chaotic.” On the interval $[\mu_3, \infty)$ the set of x that do not eventually iterate to a point in $[\mu_1, \mu_3]$ is conjectured to be of measure zero. The point μ_3 is proved to be a “homoclinic point,” in the sense that for any $\epsilon > 0$ the iterates of $[\mu_3, \mu_3 + \epsilon)$ cover the whole interval (μ_1, ∞) . It is shown that any nontrivial cycle of the $3x + 1$ function on the positive integers would be an attracting periodic orbit of f .

37. Busiso P. Chisala (1994), *Cycles in Collatz Sequences*, Publ. Math. Debrecen **45** (1994), 35–39. (MR 95h:11019).

The author shows that for any m -cycle of the Collatz map on positive *rational*s, the least element is at least as large as $(2^{\lceil m\theta \rceil/m} - 3)^{-1}$, where $\theta = \log_2 3$. Using this result, he derives a lower bound for $3x + 1$ cycle lengths based on the continued fraction of $\theta = [1, a_1, a_2, a_3, \dots]$, in which the n -th convergent is $\frac{p_n}{q_n}$, and the intermediate convergent denominator q_n^i is $iq_{n+1} + q_n$ for $0 \leq i < a_{n+1}$. If the $3x + 1$ conjecture is true for $1 \leq n \leq N$, and $N \geq (2^{C(i,k)} - 3)^{-1}$, where $C(i, k) = \frac{\lceil q_k^i \theta \rceil}{q_k^i}$, then there are no nontrivial cycles of the $3x + 1$ function on \mathbb{Z}^+ containing less than q_k^{i+1} odd terms. Using the known bound $N = 2^{40} \doteq 1.2 \times 10^{12}$, the author shows that there are at least $q_{15} = 10\,787\,915$ odd terms in any cycle of the $3x + 1$ function on \mathbb{Z}^+ . [Compare these results with those of Eliahou (1993).]

38. Dean Clark (1995), *Second-Order Difference Equations Related to the Collatz $3n + 1$ Conjecture*, J. Difference Equations & Appl., **1** (1995), 73–85. (MR 96e:11031).

The paper studies the integer-valued recurrence $\frac{x_{n+1} + x_n}{2}$ if $x_{n+1} + x_n$ is even, and $x_n = \frac{b|x_{n+1} - x_n| + 1}{2}$ if $x_{n+1} + x_n$ is odd, for $b \geq 1$ an odd integer. For $b = 1, 3, 5$ all recurrence sequences stabilize at some fixed point depending on x_1 and x_2 , provided that $x_1 = x_2 \equiv \frac{b+1}{2} \pmod{b}$. For $b \geq 7$ there exist unbounded trajectories, and periodic trajectories of period ≥ 2 . In the “convergent” cases $b = 3$ or 5 the iterates exhibit an

interesting phenomenon, which the author calls *digital convergence*, where the low order digits in base b of x_n successively stabilize before the high order bits stabilize.

39. Dean Clark and James T. Lewis (1995), *A Collatz-Type Difference Equation*, Proc. Twenty-sixth International Conference on Combinatorics, Graph Theory and Computing (Boca Raton 1995), Congr. Numer. **111** (1995), 129-135. (MR 98b:11008).

This paper studies the difference equation

$$x_n = \begin{cases} \frac{x_{n-1} + x_{n-2}}{2} & \text{if } x_{n-1} + x_{n-2} \text{ is even,} \\ x_{n-1} - x_{n-2} & \text{if } x_{n-1} + x_{n-2} \text{ is odd.} \end{cases}$$

with integer initial conditions (x_0, x_1) . It suffices to treat the case that $\gcd(x_0, x_1) = 1$. For such initial conditions the recurrence is shown to always converge to one of the 1-cycles 1 or -1 or to the 6-cycle $\{3, 2, -1, -3, -2, -1\}$.

40. Dean Clark and James T. Lewis (1998), *Symmetric solutions to a Collatz-like system of Difference Equations*, Proc. Twenty-ninth International Conference on Combinatorics, Graph Theory and Computing (Baton Rouge 1998), Congr. Numer. **131** (1998), 101-114. (see MR 99i:00021).

This paper studies the first order system of nonlinear difference equations

$$\begin{aligned} x_{n+1} &= \lfloor \frac{x_n + y_n}{2} \rfloor \\ y_{n+1} &= y_n - x_n, \end{aligned}$$

where $\lfloor \cdot \rfloor$ is the floor function (greatest integer function). Let $T(x, y) = (\lfloor \frac{x+y}{2} \rfloor, y - x)$ be a map of the plane, noting that $T(x_n, y_n) = (x_{n+1}, y_{n+1})$. The function T is an invertible map of the plane, with inverse $S = T^{-1}$ given by $S(x, y) = (\lceil x - \frac{y}{2} \rceil, \lceil x + \frac{y}{2} \rceil)$, using the ceiling function $\lceil \cdot \rceil$. One obtains an associated linear map of the plane, by not imposing the floor function above, i.e. $\tilde{T}(x, y) = (\frac{x+y}{2}, -x + y)$. The map \tilde{T} is invertible, and for arbitrary real initial conditions (x_0, y_0) the full orbit $\{\tilde{T}^{(k)}(x_0, y_0) : -\infty < k < \infty\}$ is bounded, with all points on it being confined to an invariant ellipse. The effect of the floor function is to perturb this linear dynamics. The authors focus on the question of whether all orbits having integer initial conditions (x_0, y_0) remain bounded; however they don't resolve this question. Note that integer initial conditions imply the full orbit is integral; then invertibility implies that bounded orbits of this type must be periodic.

The difference equation given above for integer initial conditions (x_0, y_0) can be transformed to a second order nonlinear recurrence by eliminating the variable x_n , obtaining.

$$y_{n+1} := \begin{cases} \frac{3y_n + 1}{2} - y_{n-1} & \text{if } y_n \equiv 1 \pmod{2} \\ \frac{3y_n}{2} - y_{n-1} & \text{if } y_n \equiv 0 \pmod{2}. \end{cases}$$

with integer initial conditions (y_0, y_1) . They note a resemblance of this recurrence in form to the $3x + 1$ problem, and view boundedness of orbits as a (vague) analogue of the $3x + 1$ Conjecture. Experimentally they observe that all integer orbits appear to be periodic, but the period of such orbits varies erratically with the initial conditions. For example

the starting condition $(64, 0)$ for T has period 87, but that of $(65, 0)$ has period 930. They give a criterion (Theorem 1) for an integer orbit to be unbounded, but conjecture it is never satisfied.

The paper also studies properties of periodic orbits imposed by some symmetry operators. They introduce the operator $Q(x, y) := (\lfloor -x + \frac{y}{2} \rfloor, y)$, observe it is an involution $Q^2 = I$ satisfying $(TQ)^2 = I$ and $S = QTQ^{-1}$. They also introduce a second symmetry operator $U(x, y) := (-x, 1 - y)$ which is an involution $U^2 = I$ that commutes with T . These operators are used to imply some symmetry properties of periodic orbits, with respect to the line $x = y$. They also derive the result (Theorem 6): the sum of the terms y_k that are even integers in a complete period of a periodic orbit is divisible by 4; the sum of all the y_k over a cycle is strictly positive and equals the number of odd y_k that appear in the cycle.

The authors do not explore what happens to orbits of T for general real initial conditions; should all orbits remain bounded in this more general situation?

41. Thomas Cloney, Eric C. Goles and Gérard Y. Vichniac (1987), *The $3x + 1$ Problem: a Quasi-Cellular Automaton*, Complex Systems **1**(1987), 349–360. (MR 88d:68080).

The paper presents computer graphics pictures of binary expansions of $\{T^{(i)}(m) : i = 1, 2, \dots\}$ for “random” large m , using black and white pixels to represent 1 resp. 0 (mod 2). It discusses patterns seen in these pictures. There are no theorems.

42. Lothar Collatz (1986), *On the Motivation of the $(3n + 1)$ -Problem*, J. of Qufu Normal University, Natural Science Edition [Qufu shi fan da xue xue bao. Zi ran ke xue ban] **12** (1986) No. 3, 9–11 (Chinese, transcribed by Zhi-Ping Ren).

Lothar Collatz describes his interest since 1928 in iteration problems represented using associated graphs and hypergraphs. He describes the structure of such graphs for several different problems. He states that he invented the $3x + 1$ problem and publicized it in many talks. He says: “Because I couldn’t solve it I never published anything. In 1952 when I came to Hamburg I told it to my colleague Prof. Dr. Helmut Hasse. He was very interested in it. He circulated the problem in seminars and in other countries.”

Note. Lothar Collatz was part of the DMV delegation to the 1950 International Congress of Mathematicians in Cambridge, Massachusetts. There Kakutani and Ulam were invited speakers. Collatz reportedly described the problem at this time to Kakutani and others in private conversations, cf. Trigg et al (1976).

43. John H. Conway (1972), *Unpredicatable Iterations*, In: Proc. 1972 Number Theory Conference, University of Colorado, Boulder, CO. 1972, pp. 49–52. (MR 52 #13717).

This paper states the $3x + 1$ problem, and shows that a more general function iteration problem similar in form to the $3x + 1$ problem is computationally undecidable.

The paper considers functions $f : \mathbb{Z} \rightarrow \mathbb{Z}$ for which there exists a finite modulus N and rational numbers $\{a_j : 0 \leq j \leq N - 1\}$ such that

$$g(n) = a_j n \quad \text{if } n \equiv j \pmod{N}.$$

In order that the map take integers to integers it is necessary that the denominator of a_j divide $\gcd(j, N)$. The computationally undecidable question becomes: Given an $f(\cdot)$ in

this class and an input value $n = 2^k$ decide whether or not some iterate of n is a power of 2. More precisely, he shows that for any recursive function $f(n)$ there exists a choice of $g(\cdot)$ such that for each n there holds $2^{f(n)} = g^k(2^n)$ for some $k \geq 1$ and this is the smallest value of k for which the iterate is a power of 2. It follows that there is no decision procedure to recognize if the iteration of such a function, starting from input $n = 2^j$, will ever encounter another power of 2, particularly whether it will encounter the value 1.

The proof uses an encoding of computations in the exponents e_p of a multiplicative factorization $n = 2^{e_2} \cdot 3^{e_3} \cdot \dots$, in which only a fixed finite number of exponents (e_2, e_3, \dots, e_{p_r}) control the computation, corresponding to the primes dividing the numerators and denominators of all a_j . The computation is based on a machine model with a finite number registers storing integers of arbitrary size, (the exponents (e_2, e_3, \dots, e_{p_r})) and there is a finite state controller. These are called *Minsky machines* in the literature, and are described in Chapter 11 of M. Minsky, *Computation: Finite and Infinite Machines*, Prentice-Hall: Englewood Cliffs, NJ 1967 (especially Sec. 11.1).

Conway (1987) later formalized this computational model as FRACTRAN, and also constructed a universal function $f(\cdot)$. See Burckel (1994) for other undecidability results.

44. John H. Conway (1987), *FRACTRAN- A Simple Universal Computing Language for Arithmetic*, in: *Open Problems in Communication and Computation* (T. M. Cover and B. Gopinath, Eds.), Springer-Verlag: New York 1987, pp. 3–27. (MR 89c:94003).

FRACTRAN is a method of universal computation based on Conway’s (1972) earlier analysis in “Unpredictable Iterations.” Successive computations are done by multiplying the current value of the computation, a positive integer, by one of a finite list of fractions, according to a definite rule which guarantees that the resulting value is still an integer. A FRACTRAN program iterates a function $g(\cdot)$ of the form

$$g(m) := \frac{p_r}{q_r} m \quad \text{if } m \equiv r \pmod{N}, \quad 0 \leq r \leq N-1,$$

and each fraction $\frac{p_r}{q_r}$ is positive with denominator q_r dividing $\gcd(N, r)$, so that the function takes positive integers as positive integers. If the input integer is of the form $m = 2^n$ then the FRACTRAN program is said to *halt* at the first value encountered which is again a power of 2, call it $2^{f(n)}$. The output $f(n) = *$ is undefined if the program never halts. A FRACTRAN program is regarded as computing the partial recursive function $\{f(n) : n \in \mathbb{Z}_{>0}\}$. FRACTRAN programs can compute any partial recursive function. The paper gives a number of examples of FRACTRAN programs, e.g. for computing the decimal digits of π , and for computing the successive primes. The prime producing algorithm was described earlier in Guy (1983b).

Section 11 of the paper discusses generalizations of the $3x+1$ problem encoded as FRACTRAN programs, including a fixed such function for which the halting problem is undecidable.

45. H. S. M. Coxeter (1971), *Cyclic Sequences and Frieze Patterns*, (*The Fourth Felix Behrend Memorial Lecture*), *Vinculum* **8** (1971), 4–7.

This lecture was given at the University of Melbourne in 1970. In this written version of the lecture, Coxeter discusses various integer sequences, including the Lyness

iteration $u_{n+1} = \frac{1+u_n}{u_{n-1}}$, which has the orbit $(1, 1, 2, 3, 2)$ as one solution. He observes that a general solution to the Lyness iteration can be produced by a frieze pattern with some indeterminates. He then introduces the $3x + 1$ iteration as "a more recent piece of mathematical gossip." He states the $3x + 1$ conjecture and then says: "I am tempted to follow the example of Paul Erdős, who offers prizes for the solutions of certain problems. If the above conjecture is true, I will gladly offer a prize of fifty dollars to the first person who send me a proof that I can understand. If it is false, I offer a prize of a hundred dollars to the first one who can establish a counterexample. I must warn you not to try this in your heads or on the back of an old envelope, because the result has been tested with an electronic computer for all $x_1 \leq 500,000$."

Note. Based on this talk, Coxeter was credited in Ogilvy (1972) with proposing the $3x + 1$ problem. For more on Frieze patterns, see H. S. M. Coxeter, *Frieze patterns*, Acta Arithmetica **18** (1971), 297–310, and J. H. Conway and H. S. M. Coxeter *Triangulated polygons and Frieze patterns*, Math. Gazette **57** (1973) no. 400, 87–94; no 401, 175–183. Vinculum is the journal of the Mathematical Association of Victoria (Melbourne, Australia).

46. Richard E. Crandall (1978), *On the "3x + 1" problem*, Math. Comp. **32** (1978), 1281–1292. (MR 58 #494).

This paper studies iteration of the " $3x+1$ " map and more generally the " $qx + r$ " map

$$T_{q,r}(x) = \begin{cases} \frac{qx+r}{2} & \text{if } x \equiv 1 \pmod{2} \\ \frac{x}{2} & \text{if } x \equiv 0 \pmod{2} \end{cases}.$$

in which $q > 1$ and $r \geq 1$ are both odd integers. He actually considers iteration of the map $C_{q,r}(\cdot)$ acting on the domain of positive odd integers, given by

$$C_{q,r}(x) = \frac{qx + r}{2^{e_2(qx+r)}},$$

where $e_2(x)$ denotes the highest power of 2 dividing x .

Most results of the paper concern the map $C_{3,1}(\cdot)$ corresponding to the $3x+1$ map. He first presents a heuristic probabilistic argument why iterates of $C_{3,1}(\cdot)$ should decrease at an exponential rate, based on this he formulates a conjecture that the number of steps $H(x)$ starting from x needed to reach 1 under iteration of $C_{3,1}(\cdot)$ should be approximately $H(x) \approx \frac{\log x}{\log \frac{16}{9}}$ for most integers. He proves that the number of odd integers n taking exactly h steps to reach 1 is at least $\frac{1}{h!}(\log_2 x)^h$. He deduces that the function $\pi_1(x)$ which counts the number of odd integers below x that eventually reach 1 under iteration of $C_{3,1}(\cdot)$ has $\pi_1(x) > x^c$ for a positive constant c . (He does not compute its value, but his proof seems to give $c = 0.05$.) He shows there are no cycles of length less than 17985 aside from the trivial cycle, using approximations to $\log_2 3$.

Concerning the " $qx + r$ " problem, he formulates the conjecture that, aside from $(q, r) = (3, 1)$, every map $C_{q,r}(\cdot)$ has at least one orbit that never visits 1. He proves that this conjecture is true whenever $r \geq 3$, and in the remaining case $r = 1$ he proves it for $q = 5$, $q = 181$ and $q = 1093$. For the first two cases he exhibits a periodic orbit not containing 1, while for $q = 1093$ he uses the fact that there are no numbers of height 2 above 1,

based on the congruence $2^{q-1} \equiv 1 \pmod{q^2}$. (This last argument would apply as well to $q = 3511$.) He argues the conjecture is true in the remaining cases because a heuristic probabilistic argument suggests that for each $q \geq 5$ the “ $qx + 1$ ” problem should have a divergent trajectory.

47. J. Leslie Davison (1977), *Some Comments on an Iteration Problem*, Proc. 6-th Manitoba Conf. On Numerical Mathematics, and Computing (Univ. of Manitoba-Winnipeg 1976), Congressus Numerantium XVIII, Utilitas Math.: Winnipeg, Manitoba 1977, pp. 55–59. (MR 58 #31773).

The author considers iteration of the map $f : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ given by

$$f(n) = \begin{cases} \frac{3n+1}{2} & \text{if } n \equiv 1 \pmod{2}, n > 1 \\ \frac{n}{2} & \text{if } n \equiv 0 \pmod{2} \\ 1 & \text{if } n = 1, \end{cases}$$

This is essentially the $3x + 1$ function, except for $n = 1$. He calls a sequence of iterates of this map a *circuit* if it starts with an odd number n , produces a sequence of odd numbers followed by a sequence of even numbers, ending at an odd number n^* . A circuit is a *cycle* if $n = n^*$.

Based on computer evidence, he conjectures that the number of circuits required during the iteration of a number n to 1 is at most $K \log n$, for some absolute constant K . He presents a probabilistic heuristic argument in support of this conjecture.

He asks whether a circuit can ever be a cycle. He shows that this question can be formulated as the exponential Diophantine equation: There exists a circuit that is a cycle if and only if there exist positive integers (k, l, h) satisfying

$$(2^{k+l} - 3^k)h = 2^l - 1.$$

(Here there are k odd numbers and l even numbers in the cycle, and $h = \frac{n+1}{2^k}$ where n is the smallest odd number in the cycle.) The trivial cycle $\{1, 2\}$ of the $3x + 1$ map corresponds to the solution $(k, l, h) = (1, 1, 1)$, Davison states he has been unable to find any other solutions. He notes that the $5x + 1$ problem has a circuit that is a cycle.

Steiner (1978) subsequently showed that $(1, 1, 1)$ is the only positive solution to the exponential Diophantine equation above.

48. Philippe Devienne, Patrick Lebègue, Jean-Christophe Routier (1993), *Halting Problem of One Binary Horn Clause is Undecidable*, *Proceedings of STACS 1993*, Lecture Notes in Computer Science No. **665**, Springer-Verlag 1993, pp. 48–57. (MR 95e:03114).

The halting problem for derivations using a single binary Horn clause for reductions is shown to be undecidable, by encoding Conway’s undecidability result on iterating periodically linear functions having no constant terms, cf. Conway (1972). In contrast, the problem of whether or not ground can be reached using reductions by a single binary Horn clause is decidable. [M. Schmidt-Schauss, *Theor. Comp. Sci.* **59** (1988), 287–296.]

49. James M. Dolan, Albert F. Gilman and Shan Manickam (1987), *A generalization of Everett's result on the Collatz $3x + 1$ problem*, Adv. Appl. Math. **8** (1987), 405–409. (MR 89a:11018).

This paper shows that for any $k \geq 1$, the set of $m \in \mathbb{Z}^+$ having k distinct iterates $T^{(i)}(m) < m$ has density one.

50. Richard Dunn (1973), *On Ulam's Problem*, Department of Computer Science, University of Colorado, Boulder, Technical Report CU-CS-011-73, 15pp.

This report gives early computer experiments on the $3x + 1$ problem. The computation numerically verifies the $3x + 1$ conjecture on a CDC 6400 computer up to 22,882,247. Dunn also calculates the densities $F(k)$ defined in equation (2.16) of Lagarias (1985) for $k \leq 21$.

51. Peter Eisele and Karl-Peter Hadeler (1990), *Game of Cards, Dynamical Systems, and a Characterization of the Floor and Ceiling Functions*, Amer. Math. Monthly **97** (1990), 466–477. (MR 91h:58086).

This paper studies iteration of the mappings $f(x) = a + \lceil \frac{x}{b} \rceil$ on \mathbb{Z} where a, b are positive integers. These are periodical linear functions (mod b). For $b \geq 2$, every trajectory becomes eventually constant or reaches a cycle of order 2.

52. Shalom Eliahou (1993), *The $3x + 1$ problem: New Lower Bounds on Nontrivial Cycle Lengths*, Discrete Math., **118**(1993), 45–56. (MR 94h:11017).

The author shows that any nontrivial cycle on \mathbb{Z}^+ of the $3x + 1$ function $T(x)$ has period $p = 301994A + 17087915B + 85137581C$ with A, B, C nonnegative integers where $B \geq 1$, and at least one of A or C is zero. Hence the minimal possible period length is at least 17087915. The method uses the continued fraction expansion of $\log_2 3$, and the truth of the $3x + 1$ Conjecture for all $n < 2^{40}$. The paper includes a table of partial quotients and convergents to the continued fraction of $\log_2 3$.

53. Peter D. T. A. Elliott (1985), *Arithmetic Functions and Integer Products*, Springer-Verlag, New York 1985. (MR 86j:11095)

An *additive function* is a function with domain \mathbb{Z}^+ , which satisfies $f(ab) = f(a) + f(b)$ if $(a, b) = 1$. In Chapters 1–3 Elliott studies additive functions having the property that $|f(an + b) - f(An + B)| \leq c_0$ for all $n \geq n_0$, for fixed positive integers a, b, A, B with $\det \begin{bmatrix} a & b \\ A & B \end{bmatrix} \neq 0$, and deduces that $|f(n)| \leq c_1(\log n)^3$. For the special case $\begin{bmatrix} a & b \\ A & B \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ an earlier argument of Wirsching yields a bound $|f(n)| \leq c_2(\log n)$. On page 19 Elliott indicates that the analogue of Wirsching's argument for $\begin{bmatrix} a & b \\ A & B \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$ leads to the $3x + 1$ function, and implies that $|f(n)| \leq c_3\sigma_\infty(n)$. A strong form of the $3x + 1$ Conjecture claims that $\sigma_\infty(n) \leq c_4 \log n$, see Lagarias and Weiss (1992). Elliott proves elsewhere by other arguments that in fact $|f(n)| \leq c_4 \log n$ holds. [P. D. T. A. Elliott, J. Number Theory **16** (1983), 285–310.]

54. C. J. Everett (1977), *Iteration of the number theoretic function $f(2n) = n, f(2n + 1) = 3n + 2$* , *Advances in Math.* **25** (1977), 42–45. (MR 56#15552).

This is one of the first research papers specifically on the $3x + 1$ function. Note that $f(\cdot)$ is the $3x + 1$ -function $T(\cdot)$. The author shows that the set of positive integers n having some iterate $T^{(k)}(n) < n$ has natural density one. The result was obtained independently and contemporaneously by Terras (1976).

55. Carolyn Farruggia, Michael Lawrence and Brian Waterhouse (1996), *The elimination of a family of periodic parity vectors in the $3x + 1$ problem*, *Pi Mu Epsilon J.* **10** (1996), 275–280.

This paper shows that the parity vector 10^k is not the parity vector of any integral periodic orbit of the $3x + 1$ mapping whenever $k \geq 2$. (For $k = 2$ the orbit with parity vector 10 is the integral orbit $\{1, 2\}$.)

56. Marc R. Feix, Amador Muriel, Danilo Merlini, and Remiglio Tartini (1995), *The $(3x+1)/2$ Problem: A Statistical Approach*, in: *Stochastic Processes, Physics and Geometry II*, Locarno 1991. (Eds: S. Albeverio, U. Cattaneo, D. Merlini) World Scientific, 1995, pp. 289–300.

This paper formulates heuristic stochastic models imitating various behaviors of the $3x + 1$ function, and compares them to some data on the $3x + 1$ function. In Sect. 2 it describes a random walk model imitating "average" behavior of forward iterates of the $3x + 1$ function. In Sect. 3 it examines trees of inverse iterates of this function, and predicts that the number of leaves at level k of the tree should grow approximately like $A(\frac{4}{3})^k$ as $k \rightarrow \infty$. In Sect. 4 it describes computer methods for rapid testing of the $3x + 1$ Conjecture. In Sect. 5 it briefly considers related functions. In particular, it considers the $3x - 1$ function and the function

$$\tilde{T}(x) = \begin{cases} \frac{x}{3} & \text{if } x \equiv 0 \pmod{3} \\ \frac{2x+1}{3} & \text{if } x \equiv 1 \pmod{3} \\ \frac{7x+1}{3} & \text{if } x \equiv 2 \pmod{3} . \end{cases}$$

Computer experiments show that the trajectories of $\tilde{T}(x)$ for $1 \leq n \leq 200,000$ all reach the fixed point $\{1\}$.

Note. Lagarias (1992) and Applegate and Lagarias (1995c) study more detailed stochastic models analogous to those given here in Sects. 2 and 3, respectively.

57. Marc R. Feix, Amador Muriel and Jean-Louis Rouet (1994), *Statistical Properties of an Iterated Arithmetic Mapping*, *J. Stat. Phys.* **76** (1994), 725–741. (MR 96b:11021).

This paper interprets the iteration of the $3x + 1$ map as exhibiting a "forgetting" mechanism concerning the iterates $(\text{mod } 2^k)$, i.e. after k iterations starting from elements it draws from a fixed residue class $(\text{mod } 2^k)$, the iterate $T^k(n)$ is uniformly distributed

$(\bmod 2^k)$. It proves that certain associated $2^k \times 2^k$ matrices M_k has $(M_k)^k = J_{2^k}$ where J_{2^k} is the doubly-stochastic $2^k \times 2^k$ matrix having all entries equal to 2^{-k} .

58. Piero Filippini (1991), *On the $3n + 1$ Problem: Something Old, Something New*, *Rendiconti di Mathematica*, Serie VII, Roma **11** (1991), 85–103. (MR 92i:11031).

This paper derives by elementary methods various facts about coalescences of trajectories and divergent trajectories. For example, the smallest counterexample n_0 to the $3x+1$ Conjecture, if one exists, must have $n_0 \equiv 7, 15, 27, 31, 39, 43, 63, 75, 79, 91 \pmod{96}$. [The final Theorem 16 has a gap in its proof, because formula (5.11) is not justified.]

59. Leopold Flatto (1992), *Z-numbers and β -transformations*, in: *Symbolic dynamics and its applications (New Haven, CT, 1991)*, *Contemp. Math.* Vol. 135, American Math. Soc., Providence, RI 1992, 181–201. (MR94c:11065).

This paper concerns the Z -number problem of Mahler (1968). A real number x is a Z -number if $0 \leq \{x(\frac{3}{2})^n\} < \frac{1}{2}$ holds for all $n \geq 0$, where $\{x\}$ denotes the fractional part of x . Mahler showed that there is at most one Z -number in each unit interval $[n, n+1)$, for positive integer n , and bounded the number of such $1 \leq n \leq X$ that can have a Z -number by $X^{0.7}$. This paper applies the β -transformation of W. Parry [Acta Math. Acad. Sci. Hungar. **11** (1960), 401–416] to get an improved upper bound on the number of n for which a Z -number exists. It studies symbolic dynamics of this transformation for $\beta = \frac{3}{2}$, and deduces that the number of $1 \leq n \leq X$ such that there is a Z -number in $[n, n+1)$ is at most X^θ with $\theta = \log_2(3/2) \approx 0.59$. The paper also obtains related results for more general Z -numbers associated to fractions $\frac{p}{q}$ having $q < p < q^2$.

60. Zachary M. Franco (1990), *Diophantine Approximation and the $qx + 1$ Problem*, Ph.D. Thesis, Univ. of Calif. at Berkeley 1990. (H. Helson, Advisor).

This thesis considers iteration of the $qx + 1$ function defined by $C_q(x) = \frac{qx+1}{2^{\text{ord}_2(qx+1)}}$, where $2^{\text{ord}_2(y)} || y$, and both q and x are odd integers. The first part of the thesis studies a conjecture of Crandall (1978), and the results appear in Franco and Pomerance (1995). The second part of the thesis gives a method to determine for a fixed q whether there are any orbits of period 2, i.e. solutions of $C_q^{(2)}(x) = x$, and it shows that for $|q| < 10^{11}$, only $q = \pm 1, \pm 3, 5, -11, -91$, and 181 have such orbits. The method uses an inequality of F. Beukers, [Acta Arith. **38** (1981) 389–410].

61. Zachary Franco and Carl Pomerance (1995), *On a Conjecture of Crandall Concerning the $qx + 1$ Problem*, *Math. Comp.* **64** (1995), 1333–1336. (MR9 5j:11019).

This paper considers iterates of the $qx+1$ function $C_q(x) = \frac{qx+1}{2^{\text{ord}_2(qx+1)}}$, where $2^{\text{ord}_2(y)} || y$ and both q and x are odd integers. Crandall (1978) conjectured that for each odd $q \geq 5$ there is some $n > 1$ such that the orbit $\{C_q^{(k)}(n) : k \geq 0\}$ does not contain 1, and proved it for $q = 5, 181$ and 1093. This paper shows that $\{q : \text{Crandall's conjecture is true for } q\}$ has asymptotic density 1, by showing the stronger result that the set $\{q : C^{(2)}(m) \neq 1 \text{ for all } m \in \mathbb{Z}\}$ has asymptotic density one.

62. David Gale (1991), *Mathematical Entertainments: More Mysteries*, *Mathematical Intelligencer* **13**, No. 3, (1991), 54–55.

This paper discusses the possible undecidability of the $3x + 1$ Conjecture, and also whether the orbit containing 8 of the original Collatz function

$$U(n) = \begin{cases} \frac{3}{2}n & \text{if } n \equiv 0 \pmod{2} \\ \frac{3}{4}n + \frac{1}{4} & \text{if } n \equiv 1 \pmod{4} \\ \frac{3}{4}n - \frac{1}{4} & \text{if } n \equiv 3 \pmod{4} \end{cases}$$

is infinite.

63. Guo-Gang Gao (1993), *On consecutive numbers of the same height in the Collatz problem*, Discrete Math., **112** (1993), 261–267. (MR 94i:11018).

This paper proves that if there exists one k -tuple of consecutive integers all having the same height and same total stopping time, then there exists infinitely many such k -tuples. (He attributes this result to P. Penning.) There is a 35654-tuple starting from $2^{500} + 1$. He conjectures that the set $\{n : C^{(k)}(n) = C^{(k)}(n + 1) \text{ for some } k \leq \log_2 n\}$ has natural density one, and proves that it has a natural density which is at least 0.389.

64. Manuel V. P. Garcia and Fabio A. Tal (1999), *A note on the generalized $3n + 1$ problem*, Acta Arith. **90**, No. 3 (1999), 245–250. (MR 2000i:11019).

This paper studies the generalized $3x + 1$ function, defined for $m > d \geq 2$ by

$$H(x) = \begin{cases} \frac{x}{d} & \text{if } x \equiv 0 \pmod{d}, \\ \frac{mx - \pi(ma)}{2} & \text{if } x \equiv a \pmod{d}, a \neq 0, \end{cases}$$

where $\pi(x)$ denotes projection (mod d) onto a fixed complete set of residues (mod d). The *Banach density* of a set $B \subset \mathbb{Z}^+$ is

$$\rho_b(B) = \limsup_{n \rightarrow \infty} \left(\max_{a \in \mathbb{Z}^+} \frac{\#(B \cap \{a, a + 1, \dots, a + n - 1\})}{n} \right).$$

The Banach density is always defined and is at least as large as the natural density of the set B , if it exists. Call two integers m_1 and m_2 *equivalent* if there is some positive integer k such that $H^{(k)}(m_1) = H^{(k)}(m_2)$. The authors assume that $m < d^{d/d-1}$, a hypothesis which implies that almost all integers have some iterate which is smaller, and which includes the $3x + 1$ function as a special case. They prove that if \mathcal{P} is any complete set of representatives of equivalence classes of \mathbb{Z}^+ then the Banach density of \mathcal{P} is zero. As a corollary they conclude that the Banach density of the orbit of any integer n under such a map H is zero. In particular, the Banach density of any divergent trajectory for such a map is zero.

65. Martin Gardner (1972), *Mathematical Games*, Scientific American **226** (1972), 114–118.

This article is one of the first places the $3x + 1$ problem is stated in print. Gardner attributes the problem to a technical report issued by M. Beeler, R. Gosper, and R. Schroepel, HAKMEM, Memo 239, Artificial Intelligence Laboratory, M.I.T., 1972, p. 64. The $3x + 1$ problem certainly predates this memo of Beeler et al., which is a collection of problems.

66. Lynn E. Garner (1981), *On the Collatz $3n + 1$ algorithm*, Proc. Amer. Math. Soc. **82** (1981), 19–22. (MR 82j:10090).

The *coefficient stopping time* $\kappa(n)$ introduced by Terras (1976) is the least iterate k such that $T^{(k)}(n) = \alpha(n)n + \beta(n)$, with $\alpha(n) < 1$. Here $\alpha(n) = \frac{3^{a(n)}}{2^n}$ where $a(n)$ is the number of iterates $T^{(j)}(n) \equiv 1 \pmod{2}$ with $0 \leq j < k$. One has $\kappa(n) \leq \sigma(n)$, where $\sigma(n)$ is the stopping time of n , and the Coefficient Stopping Time Conjecture of Terras (1976) asserts that $\kappa(n) = \sigma(n)$ for all $n \geq 2$. This paper proves that $\kappa(n) < 105,000$ implies that $\kappa(n) \leq \sigma(n)$. The proof methods used are those of Terras (1976), who proved the conjecture holds for $\kappa(n) < 2593$. They involve the use of the continued fraction expansion of $\log_2 3$ and the truth of the $3x + 1$ Conjecture for $n < 2.0 \times 10^9$.

67. Lynn E. Garner (1985), *On heights in the Collatz $3n + 1$ problem*, Discrete Math. **55** (1985), 57–64. (MR 86j:11005).

This paper shows that infinitely many pairs of consecutive integers have equal (finite) heights and equal total stopping times. To do this he studies how trajectories of consecutive integers can coalesce. Given two consecutive integers $m, m + 1$ having $T^{(i)}(m) \neq T^{(i)}(m + 1)$ for $i < k$ and $T^{(k)}(m) = T^{(k)}(m + 1)$, associate to them the pair $(\mathbf{v}, \mathbf{v}')$ of $0 - 1$ vectors of length k encoding the parity of $T^{(i)}(m)$ (resp. $T^{(i)}(m + 1)$) for $0 \leq i \leq k - 1$. Call the set \mathcal{A} of pairs $(\mathbf{v}, \mathbf{v}')$ obtained this way *admissible pairs*. Garner exhibits collections \mathcal{B} and \mathcal{S} of pairs of equal-length $0 - 1$ vectors $(\mathbf{b}, \mathbf{b}')$ and $(\mathbf{s}, \mathbf{s}')$ called *blocks* and *strings*, respectively, which have the properties: If $(\mathbf{v}, \mathbf{v}') \in \mathcal{A}$ and $(\mathbf{b}, \mathbf{b}') \in \mathcal{B}$ then the concatenated pair $(\mathbf{bv}, \mathbf{b'v}') \in \mathcal{A}$, and if $(\mathbf{s}, \mathbf{s}') \in \mathcal{S}$ then $(\mathbf{sv'}, \mathbf{s'v}) \in \mathcal{A}$. Since $(001, 100) \in \mathcal{A}$, $(10, 01) \in \mathcal{B}$ and $(000011, 101000) \in \mathcal{S}$, the set \mathcal{A} is infinite. He conjectures that: (1) a majority of all positive integers have the same height as an adjacent integer (2) arbitrarily long runs of integers of the same height occur.

68. H. Glaser and Hans-Georg Weigand (1989), *Das-ULAM Problem-Computergestützte Entdeckungen*, DdM (Didaktik der Mathematik) **17**, No. 2 (1989), 114–134.

This paper views the $3x + 1$ problem as iterating the Collatz function, and views it as an algorithmic problem between mathematics and computer science. It views study of this problem as useful as training in exploration of mathematical ideas. It formulates exploration as a series of questions to ask about it, and answers some of them. Some of these concern properties of the trees of inverse iterates of the Collatz function starting from a given number. It proves branching properties of the trees via congruence properties modulo powers of 3. This paper also discusses programming the $3x + 1$ iteration in the programming languages Pascal and LOGO.

69. Gaston Gonnet (1991), *Computations on the $3n + 1$ Conjecture*, MAPLE Technical Newsletter **0**, No. 6, Fall 1991.

This paper describes how to write computer code to efficiently compute $3x + 1$ function iterates for very large x using MAPLE. It displays a computer plot of the total stopping function for $n < 4000$, revealing an interesting structure of well-spaced clusters of points.

70. Richard K. Guy (1981), *Unsolved Problems in Number Theory*, Springer-Verlag, New York 1981.

The $3x + 1$ Problem is discussed as problem E16.

Note. Richard Guy [private conversation] informed me that he first heard of the problem in the early 1960's from his son Michael Guy, who was a student at Cambridge University and friends with John Conway. John Conway [private conversation] confirms that he heard of the problem and worked on it as a Cambridge undergraduate (BA 1959).

71. Richard K. Guy (1983a) *Don't try to solve these problems!*, Amer. Math. Monthly **90** (1983), 35–41.

The article gives some brief history of work on the $3x + 1$ problem. It mentions at second hand a statement of P. Erdős regarding the $3x + 1$ problem: “Mathematics is not yet ripe enough for such questions.”

The $3x + 1$ problem is stated as Problem 2. Problem 3 concerns cycles in the original Collatz problem, for which see Klamkin (1963). Problem 4 asks the question : Let S be the smallest set of positive integers containing 1 which is closed under $x \mapsto 2x$, $x \mapsto 3x+2$, $x \mapsto 6x + 3$. Does this set have a positive lower density?

72. Richard K. Guy (1983b), *Conway's prime producing machine*, Math. Magazine **56** (1983), 26–33. (MR 84j:10008).

This paper gives a function $g(\cdot)$ of the type in Conway (1972) having the following property. If p_j denotes the j -th prime, given in increasing order, then starting from the value $n = 2^{p_j}$ and iterating under $g(\cdot)$, the first power of 2 that is encountered in the iteration is $2^{p_{j+1}}$.

He shows that the associated register machine uses only four registers. See also the paper Conway (1987) on FRACTRAN.

73. Richard K. Guy (1986), *John Isbell's Game of Beanstalk and John Conway's Game of Beans Don't Talk*, Math. Magazine **59** (1986), 259–269. (MR 88c:90163).

John Isbell's game of Beanstalk has two players alternately make moves using the rule

$$n_{i+1} = \begin{cases} \frac{n_i}{2} & \text{if } n_i \equiv 0 \pmod{2}, \\ 3n_i \pm 1 & \text{if } n_i \equiv 1 \pmod{2}, \end{cases}$$

where they have a choice if n_i is odd. The winner is the player who moves to 1. In Conway's game the second rule becomes $\frac{3n \pm 1}{2^*}$, where 2^* is the highest power of 2 that divides the numerator. It is unknown whether or not there are positions from which neither player can force a win. If there are then the $3x + 1$ problem must have a nontrivial cycle or a divergent trajectory.

74. Lorenz Halbeisen and Norbert Hungerbühler (1997), *Optimal bounds for the length of rational Collatz cycles*, Acta Arithmetica **78** (1997), 227-239. (MR 98g:11025).

The paper presents “optimal” upper bounds for the size of a minimal element in a rational cycle of length k for the $3x + 1$ function. These estimates improve on Eliahou (1993) but currently do not lead to a better linear bound for nontrivial cycle length than the value 17,087,915 obtained by Eliahou. They show that if the $3x + 1$ Conjecture is verified for $1 \leq n \leq 212,366,032,807,211$, which is about 2.1×10^{14} , then the lower bound on cycle length jumps to 102,225,496.

[The $3x + 1$ conjecture now verified up to 1.9×10^{17} by E. Roosendaal, see the comment on Oliveira e Silva (1999).]

75. Gisbert Hasenjaeger (1990), *Hasse’s Syracuse-Problem und die Rolle der Basen*, in: *Mathesis rationis. Festschrift für Heinrich Schepers*, (A. Heinekamp, W. Lenzen, M. Schneider, Eds.) Nodus Publications: Münster 1990 (ISBN 3-89323-229-X), 329–336.

This paper, written for a philosopher’s anniversary, comments on the complexity of the $3x + 1$ problem compared to its simple appearance. It suggests looking for patterns in the iterates written in base 3 or base 27, rather than in base 2. It contains heuristic speculations and no theorems.

76. Helmut Hasse (1975), *Unsolved Problems in Elementary Number Theory*, Lectures at University of Maine (Orono), Spring 1975. Mimeographed notes.

Hasse discusses the $3x + 1$ problem on pp. 23–33. He calls it the Syracuse (or Kakutani) algorithm. He asserts that A. Fraenkel checked it for $n < 10^{50}$, which is not the case (private communication with A. Fraenkel). He states that Thompson has proved the Finite Cycles Conjecture, but this seems not to be the case, as no subsequent publication has appeared.

He suggests a generalization (mod m) for $m > d \geq 2$ and in which the map is

$$T_d(x) := \frac{mx + f(r)}{d} \text{ if } x \equiv r \pmod{d}$$

in which $f(r) \equiv -mr \pmod{d}$. He gives a probabilistic argument suggesting that all orbits are eventually periodic, when $m = d + 1$.

Hasse’s circulation of the problem motivated some of the first publications on it. He proposed a class of generalized $3x + 1$ maps studied in Möller (1978) and Heppner (1978).

77. Brian Hayes (1984), *Computer recreations: The ups and downs of hailstone numbers*, Scientific American **250**, No. 1 (1984), 10–16.

The author introduces the $3x + 1$ problem to a general audience under yet another name — hailstone numbers.

78. Ernst Heppner (1978), *Eine Bemerkung zum Hasse-Syracuse Algorithmus*, Archiv. Math. **31** (1978), 317–320. (MR 80d:10007)

This paper studies iteration of generalized $3 + 1$ maps that belong to a class formulated by H. Hasse. This class consists of maps depending on parameters of the form

$$T(n) = T_{m,d,R}(n) := \begin{cases} \frac{mn + r_j}{d} & \text{if } n \equiv j \pmod{d}, 1 \leq j \leq d-1 \\ \frac{x}{d} & \text{if } n \equiv 0 \pmod{d}. \end{cases}$$

in which the parameters (d, m) satisfy $d \geq 2$, $\gcd(m, d) = 1$, and the set $R = \{r_j : 1 \leq j \leq d-1\}$ has each $r_j \equiv -mj \pmod{d}$. The qualitative behavior of iterates of these maps are shown to depend on the relative sizes of m and d .

Heppner proves that if $m < d^{d/d-1}$ then almost all iterates get smaller, in the following quantitative sense: There exist positive real numbers δ_1, δ_2 such that for any $x > d$, and $N = \lfloor \frac{\log x}{\log d} \rfloor$, there holds

$$\#\{n \leq x : T^{(N)}(n) \geq nx^{-\delta_1}\} = O(x^{1-\delta_2}).$$

He also proves that if $m > d^{d/d-1}$ then almost all iterates get larger, in the following quantitative sense: There exist positive real numbers δ_3, δ_4 such that for any $x > d$, and $N = \lfloor \frac{\log x}{\log d} \rfloor$, there holds

$$\#\{n \leq x : T^{(N)}(n) \leq nx^{\delta_3}\} = O(x^{1-\delta_4}).$$

In these results the constants δ_j depend on d and m only while the implied constant in the O -symbols depends on m, d and R . This results improve on those of Möller (1978).

79. Stephen D. Isard and Harold M. Zwicky (1970), *Three open questions in the theory of one-symbol Smullyan systems*, SIGACT News, Issue No. 7, 1970, 11-19.

Smullyan systems are described in R. Smullyan, *First Order Logic*, [Ergebnisse der Math. Vol. 43, Springer-Verlag, NY 1968 (Corrected Reprint: Dover, NY 1995)] In this paper open question 2 concerns the set of integers that are generated by certain Smullyan systems with one symbol x . If we let n label a string of n x 's, then the system allows the two string rewriting operations $f(n) = 3n$ and $g(4m+3) = 2m+1$. The problem asks whether it is true that, starting from $n_0 = 1$, we can reach every positive number $n \equiv 1 \pmod{3}$ by a sequence of such string rewritings. The authors note that one can reach every such number except possibly those with $n \equiv 80 \pmod{81}$, and that the first few numbers in this congruence class are reachable. In the reverse direction we are allowed instead to apply either of the rules $F(3n) = n$ or $G(2m+1) = 4m+3$, and we then wish to get from an arbitrary $n \equiv 1 \pmod{3}$ to 1. An undecidability result given at the end of the paper, using Minsky machines, has some features in common with Conway (1972).

Note. This symbol rewriting problem involves many-valued functions in both the forward and backward directions, which are both linear on congruence classes to some modulus. Trigg et. al (1976) cited this problem in a discussion of the $3x+1$ problem, as part of its prehistory. The $3x+1$ problem has similar feature (cf. Everett (1977)) with the difference that the $3x+1$ function is single-valued in the forwards direction and many-valued only in the backwards direction. Michel (1993) considers other rewriting rules which are functions in one direction.

80. Frazer Jarvis (1989), *13, 31 and the $3x + 1$ problem*, Eureka **49** (1989), 22–25.

This paper studies the function $g(n) = h(n + 1) - h(n)$, where $h(n)$ is the height of n , and observes empirically that $g(n)$ appears unusually often to be representable as $13x + 31y$ with small values of x and y . It offers a heuristic explanation of this observation in terms of Diophantine approximations to $\log_6 2$. Several open problems are proposed, mostly concerning $h(n)$.

81. John A. Joseph (1998), *A chaotic extension of the Collatz function to $\mathbb{Z}_2[i]$* , Fibonacci Quarterly **36** (1998), 309–317. (MR 99f:11026).

This paper studies the function $\mathcal{F} : \mathbb{Z}_2[i] \rightarrow \mathbb{Z}_2[i]$ given by

$$\mathcal{F}(\alpha) = \begin{cases} \frac{\alpha}{2} & \text{if } \alpha \in [0] , \\ \frac{3\alpha + 1}{2} & \text{if } \alpha \in [1] , \\ \frac{3\alpha + i}{2} & \text{if } \alpha \in [i] , \\ \frac{3\alpha + 1 + i}{2} & \text{if } \alpha \in [1 + i] , \end{cases}$$

where $[\alpha]$ denotes the equivalence class of α in $\mathbb{Z}_2[i]/2\mathbb{Z}_2[i]$. The author proves that the map \tilde{T} is chaotic in the sense of Devaney [A *First Course in Dynamical Systems: Theory and Experiment*, Addison-Wesley 1992]. He shows that \tilde{T} is not conjugate to $T \times T$ via a \mathbb{Z}_2 -module isomorphism, but is topologically conjugate to $T \times T$. This is shown using an analogue \tilde{Q}_∞ of the $3x + 1$ conjugacy map Q_∞ studied in Lagarias (1985), Theorem L and in Bernstein and Lagarias (1996).

82. Frantisek Kascak (1992), *Small universal one-state linear operator algorithms*, in: Proc. MFCS '92, Lecture Notes in Computer Science No. 629, Springer-Verlag: New York 1992, pp. 327–335. [MR1255147]

A *one-state linear operator algorithm* (OLOA) with modulus m , is specified by data (a_r, b_r, c_r) a triple of integers, for each residue class $r \pmod{m}$. Given an input integer x , the OLOA does the following in one step. It finds $x \equiv r \pmod{m}$ and based on the value of r , it does the following. If $c_r = 0$, the machine halts and outputs x , calling it a final number, If $c_r \neq 0$ it computes to $(a_r x + b_r \text{ DIV } c_r = \frac{a_r x + b_r}{c_r})$, and outputs this value, provided this output is a nonnegative integer. If $\frac{a_r x + b_r}{c_r}$ is a non-integer or is negative, the the number x is called a terminal number, and the machine stops. An OLOA L computes a partially defined function $m \rightarrow f_L(m)$ as follows. For initial input value m , the OLOA is iterated, and halts with $f_L(m) = x$ if x is a final number. Otherwise $f_L(m) = \uparrow$ is viewed as undefined, and this occurs if the computation either reaches a terminal number or else runs for an infinite number of steps without stopping.

The author observes that the $3x + 1$ problem can be encoded as an OLOA L with modulus $m = 30$. The increase of modulus of the $3x + 1$ function from $m = 2$ to $m = 30$ is made in order to encode the $3x + 1$ iteration arriving at value 1 as a halting state in the sense above. The $3x + 1$ function input n is encoded as the integer $m = 5 \cdot 2^{n-1}$ to the OLOA. It is not known if the function f_L computed by this OLOA has a recursive domain $D(f_L)$, but if the $3x + 1$ conjecture is true, then $D(f_L) = \mathbb{N}$ will be recursive.

The main result of the paper is that there exists an OLOA L with modulus $m = 396$ which is a universal OLOA, i.e. there exists a unary recursive function d such that df_L is a universal unary recursive function. Here a unary partial recursive function u is *universal* if there exists some binary recursive function c such that $F_x(y) = uc(x, y)$ where $F_x(y)$ is an encoding of the x -th function in an encoding of all unary partial recursive function, evaluated at input y . In particular this machine L has an unsolvable halting problem, i.e. the function f_L has domain $D(f_L)$ which is not recursive. The encoding of the construction uses some ideas from Minsky machines, as in Conway (1972). The construction of Conway would give a universal OLOA with a much larger modulus.

83. Louis H. Kauffman (1995), *Arithmetic in the form*, Cybernetics and Systems **26** (1995), 1–57. [Zbl 0827.03033]

This interesting paper describes how to do arithmetic in certain symbolic logical systems developed by G. Spencer Brown, *Laws of Form*, George Unwin & Brown: London 1969. Spencer-Brown's formal system starts from a primitive notion of the additive void 0 and the multiplicative void 1, and uses division of space by boundaries to create numbers, which are defined using certain transformation rules defining equivalent expressions. Kauffman argues that these rules give operations more primitive than addition and multiplication. He develops a formal natural number arithmetic in this context, specifying replacement rules for rearrangements of symbolic expressions, and gives a proof of consistency of value (Theorem, p. 27).

In Appendix A he presents a second symbolic system, called *string arithmetic*. He discusses the $3x + 1$ problem in this context. This system has three kinds of symbols $*$, \langle , \rangle , where in representing integers the angles occur only in matching pairs. The integer N is represented in unary as a string of N asterisks. This integer has alternate expressions, in which the angles encode multiplication by 2. Addition is given by concatenation of expressions. The transformation rules between equivalent expressions are, if W is any string of symbols

$$\begin{aligned} ** &\longleftrightarrow \langle * \rangle \\ \rangle \langle &\longleftrightarrow (\text{blank}) \\ *W &\longleftrightarrow W* \end{aligned}$$

A natural number can be defined as any string W equivalent under these rules to some string of asterisks. For examples, these rules give as equivalent representations of $N = 4$:

$$**** = \langle * \rangle \langle * \rangle = \langle ** \rangle = \langle \langle * \rangle \rangle.$$

He observes that the $3x + 1$ problem is particularly simple to formulate in this system. He gives transformation rules defining the two steps in the $3x + 1$ function. If $N = \langle W \rangle$, then it is even, and then $\frac{N}{2} = W$, while if $N = \langle W \rangle *$, then it is odd and $\frac{3N+1}{2} = \langle W* \rangle W$. These give the transformation rules

$$(T1) : \quad \langle W \rangle \mapsto W$$

$$(T2) : \quad \langle W \rangle * \mapsto \langle W* \rangle W$$

The $3x + 1$ Conjecture is equivalent to the assertion: starting from a string of asterisks of any length, and using the transformation rules plus the rules (T1) and (T2), one can

reach the string of one asterisk *. Kauffman says: "A more mystical reason for writing the Collatz [problem] in string arithmetic is the hope that there is some subtle pattern right in the notation of string arithmetic that will show the secrets of the iteration."

84. David Kay (1972), *An Algorithm for Reducing the Size of an Integer*, Pi Mu Epsilon Journal **4** (1972), 338.

This short note proposes the $3x + 1$ problem as a possible undergraduate research project. The Collatz map is presented, being denoted $k(n)$, and it is noted that it is an unsolved problem to show that all iterates on the positive integers go to 1. It states this has been verified for all integers up to a fairly large bound.

The proposed research project asks if one can find integers p, q, r with $p, q \geq 2$, having the property that a result analogous to the $3x + 1$ Conjecture can be rigorously proved for the function

$$k(n) := \begin{cases} \frac{n}{p} & \text{if } n \equiv 0 \pmod{p} . \\ qn + r & \text{if } n \not\equiv 0 \pmod{p} . \end{cases}$$

Note. This project was listed as proposed by the Editor of Pi Mu Epsilon; the Editor was David Kay.

85. Timothy P. Keller (1999), *Finite cycles of certain periodically linear functions*, Missouri J. Math. Sci. **11** (1999), no. 3, 152–157. (MR 1717767)

This paper is motivated by the original Collatz function $f(3n) = 2n$, $f(3n + 1) = 4n + 1$, $f(3n + 2) = 4n + 3$, which is a permutation of \mathbb{Z} , see Klamkin (1963). The class of periodically linear functions which are permutations. were characterized by Venturini (1997). The author defines a *permutation of type V* to be a periodically linear function g_p defined for an odd integer $p > 1$ by $g_p(pn + r) = (p + (-1)^r)n + r$ for $0 \leq r \leq p - 1$. The permutation g_3 is conjugate to the original Collatz function. This paper shows that for each $L \geq 1$ any permutation of type V has only finitely many cycles of period L . The author conjectures that for each odd p there is a constant $s_0(p)$ such that if $L \geq s_0(p)$ is the minimal period of a periodic orbit of g_p , then L is a denominator of a convergent of the continued fraction expansion of $\gamma_p := \frac{\log(p-1) - \log p}{\log p - \log(p-1)}$. For $p = 7$ these denominators start $\Lambda(7) = \{1, 2, 13, 28, 265, 293, \dots\}$. He finds that g_7 has an orbit of period 265 with starting value $n_1 = 1621$, and an orbit of period 293 with starting value $n_2 = 293$. This question was raised for the original Collatz function by Shanks (1965).

86. Murray S. Klamkin (1963), *Problem 63 – 13**, SIAM Review **5** (1963), 275–276.

He states the problem: "Consider the infinite permutation

$$P \equiv \left(\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 & \dots \\ 1 & 3 & 2 & 5 & 7 & 4 & \dots \end{array} \right)$$

taking $n \mapsto f(n)$ where $f : \mathbb{N}^+ \rightarrow \mathbb{N}^+$ is given by $f(3n) = 2n$, $f(3n-1) = 4n-1$, $f(3n-2) = 4n-3$. We now write P as a product of cycles

$$P \equiv (1) (2, 3) (4, 5, 7, 9, 6) (8, 11, 15, \dots$$

It is conjectured that the cycle $(8, 11, 15, \dots)$ is infinite. Other problems concerning P are:

(a) Does the permutation P consist of finitely many cycles?

(b) Are there any more finite cycles than those indicated? “

This function was the original function proposed by L. Collatz in his private notes in 1932. See Shanks (1965) and Atkin (1966) for comments on this problem. This problem remains unsolved concerning the orbit of $n = 8$ and part (a). Concerning (b) one more cycle was found, of period 12 with smallest element $n = 144$. Atkin (1966) presents a heuristic argument suggesting there are finitely many cycles.

Note. In 1963 M. Klamkin proposed another problem, jointly with A. L. Tritter, concerning the orbit structure of a different infinite permutation of the integers [*Problem 5109*, Amer. Math. Monthly **70** (1963), 572–573]. For this integer permutation all orbits are cycles. A solution to this problem was given by G. Bergman, Amer. Math. Monthly **71** (1964), 569–570.

87. Ivan Korec (1992), *The $3x + 1$ Problem, Generalized Pascal Triangles, and Cellular Automata*, Math. Slovaca, **42** (1992), 547–563. (MR 94g:11019).

This paper shows that the iterates of the Collatz function $C(x)$ can actually be encoded in a simple way by a one-dimensional nearest-neighbor cellular automaton with 7 states. The automaton encodes the iterates of the function $C(x)$ written in base 6. The encoding is possible because the map $x \rightarrow 3x + 1$ in base 6 does not have carries propagate. (Compare the $C(x)$ -iterates of $x = 26$ and $x = 27$ in base 6.) The $3x + 1$ Conjecture is reformulated in terms of special structural properties of the languages output by such cellular automata.

88. Ivan Korec (1994), *A Density Estimate for the $3x + 1$ Problem*, Math. Slovaca **44** (1994), 85–89. (MR 95h:11022).

Let $S_\beta = \{n : \text{some } T^{(k)}(n) < n^\beta\}$. This paper shows that for any $\beta > \frac{\log 3}{\log 4} \doteq .7925$ the set S_β has density one. The proof follows the method of E. Heppner [Arch. Math. **31** (1978) 317–320].

89. Ivan Korec and Stefan Znam (1987), *A Note on the $3x + 1$ Problem*, Amer. Math. Monthly **94** (1987), 771–772. (MR 90g:11023).

This paper shows that to prove the $3x + 1$ Conjecture it suffices to verify it for the set of all numbers $m \equiv a \pmod{p^n}$, for any fixed $n \geq 1$, provided that 2 is a primitive root \pmod{p} and $(a, p) = 1$. This set has density p^{-n} .

90. Ilia Krasikov (1989), *How many numbers satisfy the $3x + 1$ Conjecture?*, Internatl. J. Math. & Math. Sci. **12** (1989), 791–796. (MR 91c:11013).

This paper shows that the number of integers $\leq x$ for which the $3x + 1$ function has an iterate that is 1 is at least $x^{3/7}$. More generally, if $\theta_a(x) = \{n \leq x : \text{some } T^{(k)}(n) = a\}$, then he shows that, for $a \not\equiv 0 \pmod{3}$, $\theta_a(x)$ contains at least $x^{3/7}$ elements, for large enough x . For each fixed $k \geq 2$ this paper derives a system of difference inequalities based on information $\pmod{3^k}$. The bound $x^{3/7}$ was obtained using $k = 2$, and by using larger

values of k better exponents can be obtained. This was done in Applegate and Lagarias (1995b) and Krasikov and Lagarias (2003).

91. James R. Kuttler (1994), *On the $3x+1$ Problem*, Adv. Appl. Math. **15** (1994), 183–185. (Zbl. 803.11018.)

The author states the oft-discovered fact that $T^{(k)}(2^k n - 1) = 3^k n - 1$ and derives his main result that if r runs over all odd integers $1 \leq r \leq 2^k - 1$ then $T^{(k)}(2^k n + r) = 3^p n + s$, in which $p \in \{1, 2, \dots, k\}$ and $1 \leq s \leq 3^p$, and each value of p occurs exactly $\binom{k-1}{p-1}$ times. Thus the density of integers with $T^{(k)}(n) > n$ is exactly $\frac{\alpha}{2^k}$, where $\alpha = \sum_{p > \theta k} \binom{k-1}{p-1}$, and $\theta = \log_2 3$. This counts the number of inflating vectors in Theorem C in Lagarias (1985).

92. Jeffrey C. Lagarias (1985), *The $3x+1$ problem and its generalizations*, Amer. Math. Monthly **92** (1985), 3–23. (MR 86i:11043).

This paper is a survey with extensive bibliography of known results on $3x+1$ problem and related problems up to 1984. It also contains improvements of previous results and some new results, including in particular Theorems D, E, F, L and M.

Theorem O has several misprints. The method of Conway (1972) gives, for any partial recursive function f , a periodic piecewise linear function g , with the property: iterating g with the starting value 2^n will never be a power of 2 if $f(n)$ is undefined, and will eventually reach a power of 2 if $f(n)$ is defined, and the first such power of 2 will be $2^{f(n)}$. Thus, in parts (ii) and (iii) of Theorem O, occurrences of n must be replaced by 2^n . On page 15, the thirteenth partial quotient of $\log_2(3)$ should be $q_{13} = 190537$.

An updated version of this paper is accessible on the World-Wide Web:
<http://www.cecm.sfu.ca/organics/papers/lagarias/index.html>.

93. Jeffrey C. Lagarias (1990), *The set of rational cycles for the $3x+1$ problem*, Acta Arithmetica **56** (1990), 33–53. (MR 91i:11024).

This paper studies the sets of those integer cycles of

$$T_k(x) = \begin{cases} \frac{3x+k}{2} & \text{if } x \equiv 1 \pmod{2}, \\ \frac{x}{2} & \text{if } x \equiv 0 \pmod{2}, \end{cases}$$

for positive $k \equiv \pm 1 \pmod{6}$ which have $(x, k) = 1$. These correspond to rational cycles $\frac{x}{k}$ of the $3x+1$ function T . It conjectures that every T_k has such an integer cycle. It shows that infinitely many k have at least $k^{1-\epsilon}$ distinct such cycles of period at most $\log k$, and infinitely many k have no such cycles having period length less than $k^{1/3}$. Estimates are given for the counting function $C(k, y)$ counting the number of such cycles of T_k of period $\leq y$, for all $k \leq x$ with $y = \beta \log x$. In particular $C(k, 1.01k) \leq 5k(\log k)^5$.

94. Jeffrey C. Lagarias (1999), *How random are $3x+1$ function iterates?*, in: *The Mathematician and Pied Puzzler: A Collection in Tribute to Martin Gardner*, A. K. Peters, Ltd.: Natick, Mass. 1999, pp. 253–266.

This paper briefly summarizes results on extreme trajectories of $3x+1$ iterates,

including those in Applegate and Lagarias (1995a), (1995b), (1995c) and Lagarias and Weiss (1992). It also presents some large integers n found by V. Vyssotsky whose trajectories take $c \log n$ steps to iterate to 1 for various $c > 35$. For example $n = 37\,66497\,18609\,59140\,59576\,52867\,40059$ has $\sigma_\infty(n) = 2565$ and $\gamma(n) = 35.2789$. It also mentions some unsolved problems.

The examples above were the largest values of γ for $3x + 1$ iterates known at the time, but are now superseded by examples of Roosendaal (2004+) achieving $\gamma = 36.716$.

95. Jeffrey C. Lagarias, Horacio A. Porta and Kenneth B. Stolarsky (1993), *Asymmetric Tent Map Expansions I. Eventually Periodic Points*, J. London Math. Soc., **47** (1993), 542–556. (MR 94h:58139).

This paper studies the set of eventually periodic points $\text{Per}(T_\alpha)$ of the asymmetric tent map

$$T_\alpha(x) = \begin{cases} \alpha x & \text{if } 0 < x \leq \frac{1}{\alpha} \\ \frac{\alpha}{\alpha-1}(1-x) & \text{if } \frac{1}{\alpha} \leq x < 1, \end{cases}$$

where $\alpha > 1$ is real. It shows that $\text{Per}(T_\alpha) = \mathbb{Q}(\alpha) \cap [0, 1]$ for those α such that both α and $\frac{\alpha}{\alpha-1}$ are Pisot numbers. It finds 11 such numbers, of degree up to four, and proves that the set of all such numbers is finite.

It conjectures that the property $\text{Per}(T_\alpha) = \mathbb{Q}(\alpha) \cap [0, 1]$ holds for certain other α , including the real root of $x^5 - x^3 - 1 = 0$. The problem of proving that $\text{Per}(T_\alpha) = \mathbb{Q}(\alpha) \cap [0, 1]$ in these cases appears analogous to the problem of proving that the $3x + 1$ function has no divergent trajectories.

[C. J. Smyth, *There are only eleven special Pisot numbers*, (Bull. London Math. Soc. **31** (1989) 1–5) proved that the set of 11 numbers found above is the complete set.]

96. Jeffrey C. Lagarias, Horacio A. Porta and Kenneth B. Stolarsky (1994), *Asymmetric Tent Map Expansions II. Purely Periodic Points*, Illinois J. Math. **38** (1994), 574–588. (MR 96b:58093, Zbl 809:11042).

This paper continues Lagarias, Porta and Stolarsky (1993). It studies the set $\text{Fix}(T_\alpha)$ of purely periodic points of the asymmetric tent map $T_\alpha(\cdot)$ and the set $\text{Per}_0(T_\alpha)$ with terminating T_α -expansion, in those cases when α and $\frac{\alpha}{1-\alpha}$ are simultaneously Pisot numbers. It shows that $\text{Fix}(T_\alpha) \subseteq \{\gamma \in \mathbb{Q}(\alpha) \text{ and } \sigma(\alpha) \in A_\alpha^\sigma \text{ for all embeddings } \sigma : \mathbb{Q}(\alpha) \rightarrow \mathbb{C} \text{ with } \sigma(\alpha) \neq \alpha\}$, in which each set A_α^σ is a compact set in \mathbb{C} that is the attractor of a certain hyperbolic iterated function system. It shows that equality holds in this inclusion in some cases, and not in others. Some related results for $\text{Per}_0(T_\alpha)$ are established.

97. Jeffrey C. Lagarias and Alan Weiss (1992), *The $3x + 1$ Problem: Two Stochastic Models*, Annals of Applied Probability **2** (1992), 229–261. (MR 92k:60159).

This paper studies two stochastic models that mimic the “pseudorandom” behavior of the $3x + 1$ function. The models are branching random walks, and the analysis uses the theory of large deviations.

For the models the average number of steps to get to 1 is $\alpha_0 \log n$, where $\alpha_0 = \left(\frac{1}{2} \log \frac{3}{4}\right)^{-1} \approx 6.9$. For both models it is shown that there is a constant $c_0 = 41.677\dots$ such that with probability one for any $\epsilon > 0$ only finitely many m take $(c_0 + \epsilon) \log m$ iterations to take the value 1, while infinitely many m take at least $(c_0 - \epsilon) \log m$ iterations to do so. This prediction is shown to be consistent with empirical data for the $3x + 1$ function.

The paper also studies the maximum excursion

$$t(n) := \max_{k \geq 1} T^{(k)}(n)$$

and conjectures that $t(n) < n^{2+o(1)}$ as $n \rightarrow \infty$. An analog of this conjecture is proved for one stochastic model. The conjecture $t(n) \leq n^{2+o(1)}$ is consistent with empirical data of Leavens and Vermeulen (1992) for $n < 10^{12}$.

98. Gary T. Leavens and Mike Vermeulen (1992), *3x + 1 Search Programs*, Computers & Mathematics, with Applications **24**, No. 11,(1992), 79–99. (MR 93k:68047).

This paper describes methods for computing $3x + 1$ function iterates, and gives results of extensive computations done on a distributed network of workstations, taking an estimated 10 CPU-years in total. The $3x+1$ Conjecture is verified for all $n < 5.6 \times 10^{13}$. Presents statistics on various types of extremal trajectories of $3x+1$ iterates in this range. Gives a detailed discussion of techniques used for program optimization.

[This bound for verifying the $3x + 1$ conjecture is now superseded by that of Oliveira e Silva (1999).]

99. George M. Leigh (1986), *A Markov process underlying the generalized Syracuse algorithm*, Acta Arithmetica **46** (1986), 125–143. (MR 87i:11099)

This paper considers mappings $T(x) = \frac{m_i x - r_i}{d}$ if $x \equiv i \pmod{d}$, where all $r_i \equiv im_i \pmod{d}$. This work is motivated by earlier work of Matthews and Watts (1984), (1985), and introduces significant new ideas.

Given such a mapping T and an auxiliary modulus m it introduces two Markov chains, denoted $\{X_n\}$ and $\{Y_n\}$, whose behavior encodes information on the iterates of T (modulo md^k) for all $k \geq 1$. In general, both Markov chains have a countable number of states; however in many interesting cases both these chains are finite state Markov chains. The states of each chain are labelled by certain congruence classes $B(x_j, M_j) := \{x : x \equiv x_j \pmod{M_j}\}$ with $M_j | md^k$ for some $k \geq 1$. (In particular $B(x_j, M_j) \subset B(x_k, M_k)$ may occur.) Any finite path in such a chain can be realized by a sequence of iterates $x_j = T^j(x_0)$ for some $x_0 \in \mathbb{Z}$ satisfying the congruences specified by the states. The author shows the two chains contain equivalent information; the chain $\{X_n\}$ has in general fewer states than the chain $\{Y_n\}$ but the latter is more suitable for theoretical analysis, in particular one chain is finite if and only if the other one is. If such a chain has a positive ergodic class of states, then the limiting frequency distribution of state occupation of a path in the chain in this class exists, and from this the corresponding frequencies of iterates in a given congruence class \pmod{m} can be calculated (Theorem 1- Theorem 4).

For applications to the map T on the integers, the author's guiding conjecture is: *any condition that occurs with probability zero in the Markov chain model does not occur in divergent trajectories on T* . Thus the author conjectures that in cases where the Markov

chain is finite, there are limiting densities that divergent trajectories for T (should they exist) spend in each residue class $j \pmod{m}$, obtained from frequency distribution in ergodic states of the chain.

The author gets a complete analysis whenever the associated Markov chains are finite. Furthermore Theorem 7 shows that any map T satisfying the condition $\gcd(m_i, d^2) = \gcd(m_i, d)$ for $0 \leq i \leq d-1$, has, for every modulus m , both Markov chains $\{X_n\}$ and $\{Y_n\}$ being finite chains. All maps studied in the earlier work of Matthews and Watts (1984, 1985) satisfy this condition, and the author recovers nearly all of their results in these cases.

In Sect. 5 the author suggests that when infinite Markov chains occur, that one approximate them using a series of larger and larger finite Markov chains obtained by truncation. Conjecture 2 predicts limiting frequencies of iterates \pmod{m} for the map T , even in these cases. The paper concludes with several worked examples.

This Markov chain approach was extended further by Venturini (1992).

100. Simon Letherman, Dierk Schleicher and Reg Wood (1999), *On the $3X + 1$ problem and holomorphic dynamics*, Experimental Math. **8**, No. 3 (1999), 241–251. (MR 2000g:37049.)

This paper studies the class of entire functions

$$f_h(z) := \frac{z}{2} + (z + \frac{1}{2})\left(\frac{1 + \cos \pi z}{2}\right) + \frac{1}{\pi}\left(\frac{1}{2} - \cos \pi z\right) \sin \pi z + h(z)(\sin \pi z)^2,$$

in which $h(z)$ is an arbitrary entire function. Each function in this class reproduces the $3x + 1$ -function on the integers, and the set of integers is contained in the set of critical points of the function. The simplest such function takes $h(z)$ to be identically zero, and is denoted $f_0(z)$. The authors study the iteration properties of this map in the complex plane \mathbb{C} . They show that \mathbb{Z} is contained in the Fatou set of $f_h(z)$. There is a classification of the connected components of the Fatou set of an entire function into six categories: (1) (periodic) immediate basins of (super-) attracting periodic points, (2) (periodic) immediate basins of attraction of rationally indifferent periodic points, (3) (periodic) Siegel disks, (4) periodic domains at infinity (Baker domains), (5) preperiodic components of any of the above, and (6) wandering domains. The following results all apply to the function $f_0(z)$, and some of them are proved for more general $f_h(z)$. The Fatou component of any integer must be in the basin of attraction of a superattracting periodic point or be in a wandering domain; the authors conjecture the latter does not happen. The existence of a divergent trajectory is shown equivalent to $f_0(z)$ containing a wandering domain containing some integer. No two integers are in the same Fatou component, except possibly -1 and -2 . The real axis contains points of the Julia set between any two integers, except possibly -1 and -2 . It is known that holomorphic dynamics of an entire function f is controlled by its critical points. The critical points of $f_0(z)$ on the real axis consist of the integers together with $-\frac{1}{2}$. The authors would like to choose h to reduce the number of other critical points, to simplify the dynamics. However they show that any map $f_h(z)$ must contain at least one more critical point in addition to the critical points on the real axis (and presumably infinitely many.) The authors compare and contrast their results with the real dynamics studied in Chamberland (1996).

101. Heinrich Lunkenheimer (1988), *Eine kleine Untersuchung zu einem zahlentheoretischen Problem*, PM (Praxis der Mathematik in der Schule) **30** (1988), 4–9.

The paper considers the Collatz problem, giving no prior history or references. It observes the coalescences of some orbits. It lists some geometric series of integers which iterate to 1.

102. Kurt Mahler (1968), *An unsolved problem on the powers of $3/2$* , J. Australian Math. Soc. **8** (1968), 313–321. (MR 37 #2694).

A Z -number is a real number ξ such that $0 \leq (\frac{3}{2})^k \xi \leq \frac{1}{2}$ holds for all $k \geq 1$, where x denotes the fractional part of x . Do Z -numbers exist? The Z -number problem was originally proposed by Prof. Saburo Uchiyama (Tsukuba Univ.) according to S. Ando (personal communication), and was motivated by a connection with the function $g(k)$ in Waring’s problem, for which see G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers* (4-th edition), Oxford Univ. Press 1960, Theorem 393 ff. and Stemmler (1964).

Mahler shows that existence of Z -numbers relates to a question concerning the iteration of the function

$$g(x) = \begin{cases} \frac{3x+1}{2} & \text{if } x \equiv 1 \pmod{2} . \\ \frac{3x}{2} & \text{if } x \equiv 0 \pmod{2} . \end{cases}$$

Mahler showed that a Z -number exists in the interval $[n, n+1)$ if and only if no iterate $g^{(k)}(n) \equiv 3 \pmod{4}$. He uses this relation to prove that the number of Z -numbers below x is at most $O(x^{0.7})$. He conjectures that no Z -numbers exist, a problem which is still unsolved.

Note. Leopold Flatto (1992) subsequently improved Mahler’s upper bound on Z -numbers below x to $O(x^\theta)$, with $\theta = \log_2 \frac{3}{2} \approx 0.59$.

103. Jerzy Marcinkowski (1999), *Achilles, Turtle, and Undecidable Boundedness Problems for Small DATALOG programs*, SIAM J. Comput. **29** (1999), 231–257. (MR 2002d:68035).

DATALOG is the language of logic programs without function symbols. A DATALOG program consists of a finite set of Horn clauses in the language of first order logic without equality and without functions.

The author introduces *Achilles-Turtle machines*, which model the iteration of *Conway functions*, as introduced in Conway (1972). These are functions $g : \mathbb{N} \rightarrow \mathbb{N}$ having the form

$$g(n) = \frac{a_j}{q_j} n \quad \text{if } n \equiv j \pmod{p}, \quad 0 \leq j \leq p-1,$$

where for each j , $a_j \geq 0, q_j \geq 1$ are integers with $q_j | \gcd(j, p)$ and with $\frac{a_j}{q_j} \leq p$. He cites Devienne, Lebégue and Routier (1993) for the idea of relating Conway functions to Horn clauses. In Section 2.3 he explicitly describes an Achilles-Turtle machine associated to computing the Collatz function.

The paper proves that several questions concerning the uniform boundedness of computations are undecidable. These include uniform boundedness for ternary linear programs;

uniform boundedness for single recursive rule ternary programs; and uniform boundedness of single rule programs. These results give no information regarding the $3x + 1$ function itself.

104. Dănuț Marcu (1991), *The powers of two and three*, Discrete Math. **89** (1991), 211-212. (MR 92h:11026).

This paper obtains a similar result to Narkiewicz (1980), with a similar proof, but obtains a slightly worse bound on the exceptional set $N(T) < 2.52T^\theta$ where $\theta = \frac{\log 2}{\log 3}$.

Note. This paper cites a paper of Gupta appearing in the same journal as Narkiewicz's paper [Univ. Beograd Elektrotech. Fak. Ser. Mat. Phys.] slightly earlier in the same year, but fails to cite Narkiewicz (1980). For clarification, consult the Wikipedia article on D. Marcu.

105. Maurice Margenstern and Yuri Matiyasevich (1999), *A binomial representation of the $3X + 1$ problem*, Acta Arithmetica **91**, No. 4 (1999), 367-378. (MR 2001g:11015).

This paper encodes the $3x + 1$ problem as a logical problem using one universal quantifier and existential quantifiers, with an arithmetical formula using polynomials and binomial coefficients. The authors observe that the use of such expressions in a language with binomial coefficients often leads to shorter formulations than are possible in a language just allowing polynomial equations and quantifiers. They give three equivalent restatements of the $3x + 1$ Conjecture in terms of quantified binomial coefficient equations.

106. Keith R. Matthews (1992), *Some Borel measures associated with the generalized Collatz mapping*, Colloq. Math. **53** (1992), 191-202. (MR 93i:11090).

This paper studies maps of the form

$$T(x) = \frac{m_i x - r_i}{d} \quad \text{if } x \equiv i \pmod{d}.$$

These extend first to maps on the d -adic integers \mathbb{Z}_d , and then further to maps on the polyadic integers $\hat{\mathbb{Z}}$. Here $\hat{\mathbb{Z}}$ is the projective limit of the set of homomorphisms $\phi_{n,m} : \mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ where $m|n$. The open sets $B(j, m) := \{x \in \hat{\mathbb{Z}} : x \equiv j \pmod{m}\}$ put a topology on $\hat{\mathbb{Z}}$, which has a Haar measure $\sigma(B(j, m)) = \frac{1}{m}$. The paper proves conjectures of Buttsworth and Matthews (1990) on the structure of all ergodic open sets (mod m). In particular the ergodic sets link together to give finitely many projective systems, each giving T -invariant measure on $\hat{\mathbb{Z}}$. The paper gives examples where there are infinitely many ergodic sets.

107. Keith R. Matthews and George M. Leigh (1987), *A generalization of the Syracuse algorithm to $F_q[x]$* , J. Number Theory **25** (1987), 274-278. (MR 88f:11116).

This paper defines mappings analogous to the $3x + 1$ function on polynomials over finite fields, e.g. $T(f) = \frac{f}{x}$ if $f \equiv 0 \pmod{x}$ and $\frac{(x+1)^3 f + 1}{x}$ if $f \equiv 1 \pmod{x}$, over $\text{GF}(2)$. It proves that divergent trajectories exist for certain such maps. These divergent trajectories have a regular behavior.

108. Keith R. Matthews and Anthony M. Watts (1984), *A generalization of Hasse's generalization of the Syracuse algorithm*, Acta. Arithmetica **43** (1984), 167–175. (MR 85i:11068).

This paper studies functions $T(x) = \frac{m_i x - r_i}{d}$ for $x \equiv i \pmod{d}$, where all m_i are positive integers and $r_i \equiv im_i \pmod{d}$. It is shown that if $\{T^{(k)}(m) : k \geq 0\}$ is unbounded and uniformly distributed \pmod{d} then $m_1 m_2 \cdots m_d > d^d$ and $\lim_{k \rightarrow \infty} |T^{(k)}(m)|^{1/k} = \frac{1}{d}(m_1 \cdots m_d)^{1/d}$. The function T is extended to a mapping on the d -adic integers and is shown to be strongly mixing, hence ergodic, on \mathbb{Z}_d . The trajectories $\{T^{(k)}(\omega) : k \geq 0\}$ for almost all $\omega \in \mathbb{Z}_d$ are equidistributed $\pmod{d^k}$ for all $k \geq 1$.

109. Keith R. Matthews and Anthony M. Watts (1985), *A Markov approach to the generalized Syracuse algorithm*, Acta Arithmetica **45** (1985), 29–42. (MR 87c:11071).

This paper studies the functions $T(x) = \frac{m_i x - r_i}{d}$ for $x \equiv i \pmod{d}$, where all m_i are positive integers and $r_i \equiv im_i \pmod{d}$, which were considered in Matthews and Watts (1984). Given a modulus m one associates to T a row-stochastic matrix $Q = [q_{jk}]$ in which j, k index residue classes \pmod{m} and q_{jk} equals $\frac{1}{md}$ times the number of residue classes \pmod{md} which are $\equiv k \pmod{m}$ and whose image under T is $\equiv j \pmod{m}$. It gives sufficient conditions for the entries of Q^l to be the analogous probabilities associated to the iterated mapping $T^{(l)}$. Matthews and Watts conjecture that if \mathcal{S} is an ergodic set of residues \pmod{m} and $(\alpha_i; i \in \mathcal{S})$ is the corresponding stationary vector on \mathcal{S} , and if $A = \prod_{i \in \mathcal{S}} (\frac{m_i}{d})^{\alpha_i} < 1$ then all trajectories of T starting in \mathcal{S} will eventually be periodic, while if $A > 1$ almost all trajectories starting in \mathcal{S} will diverge. Some numerical examples are given.

110. Danilo Merlini and Nicoletta Sala (1999), *On the Fibonacci's Attractor and the Long Orbits in the $3n + 1$ Problem*, International Journal of Chaos Theory and Applications **4**, No. 2-3 (1999), 75–84.

This paper studies heuristic models for the Collatz tree of inverse iterates of the Collatz function $C(x)$, which the authors call the "chalice". They also consider the length of the longest trajectories for the Collatz map. The author's predictions are that the number of leaves in the Collatz tree at depth k should grow like Ac^k where $c = \frac{1}{2}(1 + \sqrt{\frac{7}{3}}) \approx 1.2637$ and $A = \frac{3c}{3c-2} \approx 0.6545$. This estimate is shown to agree closely with numerical data computed to depth $k = 32$. They predict that the longest orbits of the Collatz map should take no more than $67.1 \log n$ steps to reach 1. This is shown to agree with empirical data for the longest orbit found up to 10^{10} by Leavens and Vermeulen (1992).

Note. These models have some features related to that in Lagarias and Weiss (1992) for the $3x + 1$ map $T(n)$. The asymptotic constants they obtain differ from those in the models of Lagarias and Weiss (1992) because the Collatz function $C(x)$ takes one extra iteration each time an odd number occurs, compared to for the $3x + 1$ mapping $T(x)$. The Applegate and Lagarias estimate for the number of nodes in the k -th level of a $3x + 1$ tree is $(\frac{4}{3})^{k(1+o(1))}$, and their estimate for the maximal number of steps to reach 1 for the $3x + 1$ map is about $41.677 \log n$.

111. Karl Heinz Metzger (1995), *Untersuchungen zum $(3n + 1)$ -Algorithmus. Teil I: Das Problem der Zyklen*, PM (Praxis der Mathematik in der Schule) **38** (1995), 255–257. (Nachtrag zum Beweis der $(3n + 1)$ -Problems, PM **39** (1996), 217.)

The author studies cycles for the $3x + 1$ map, and more generally considers the $bx + 1$ map for odd b . He exhibits a cycle for the $5x + 1$ that does not contain 1, namely $n = 13$. He obtains a general formula (4.1) for a rational number a to be in a cycle of the $3x + 1$ map, as in Lagarias (1990). At the end of the paper is a theorem asserting that the only cycle of the Collatz map acting on the positive integers is the trivial cycle $\{1, 2, 4\}$. The proof of this theorem has a gap, however. The author points this out in the addendum, and says he hopes to return to the question in future papers.

The author's approach to proving that the trivial cycle is the only cycle on the positive integers is as follows. He observes that the condition (4.1) for c to be a rational cycle with ν odd elements can be rewritten as

$$c = \frac{A_\nu + x_0^*}{A_\nu + y_0^*}$$

for certain integers (x_0^*, y_0^*) , taking $A_\nu = 2^{2\nu} - 3^\nu$. He then observes that this can be viewed as a special case of a linear Diophantine equation in (x, y) ,

$$c = \frac{A_\nu + x}{A_\nu + y}, \quad \text{with } A, c \text{ fixed,}$$

which has the general solution $(x, y) = ((c^2 - 1)A + \lambda c, (c - 1)A + \lambda c)$, for some integer λ . His proof shows that the case where (x_0^*, y_0^*) has associated value $\lambda = 0$ leads to a contradiction. The gap in the proof is that cases where $\lambda \neq 0$ are not ruled out.

112. Karl Heinz Metzger (1999), *Zyklenbestimmung beim $(bn + 1)$ -Algorithmus*, PM (Praxis der Mathematik in der Schule) **41** (1999), 25–27.

For an odd number b the Collatz version of the $bx + 1$ function is

$$C_b(x) = \begin{cases} bx + 1 & \text{if } x \equiv 1 \pmod{2}, \\ \frac{x}{2} & \text{if } x \equiv 0 \pmod{2}. \end{cases}$$

The author studies cycles for the $bn + 1$ map. Let the *size* of a cycle count the number of distinct odd integers it contains. The author first observes that 1 is in a size one cycle for the $bn + 1$ problem if and only if $b = 2^\nu - 1$ for some $\nu \geq 1$. For the $5x + 1$ problem he shows that on the positive integers there is no cycle of size 1, a unique cycle of size 2, having smallest element $n = 1$, and exactly two cycles of size 3, having smallest elements $n = 13$ and $n = 17$, respectively.

At the end of the paper the author states a theorem asserting that the only cycle of the Collatz map acting on the positive integers is the trivial cycle $\{1, 2, 4\}$. This paper apparently is intended to repair the faulty proof in Metzger (1995). However the proof of this result still has a gap.

113. Pascal Michel (1993), *Busy Beaver Competition and Collatz-like Problems*, Archive Math. Logic **32** (1993), 351–367. (MR 94f:03048).

The Busy Beaver problem is to find that Turing machine which, among all k -state

Turing machines, when given the empty tape as input, eventually halts and produces the largest number $\Sigma(k)$ of ones on the output tape. The function $\Sigma(k)$ is well-known to be non-recursive. This paper shows that the current Busy Beaver record-holder for 5-state Turing machine computes a Collatz-like function. This machine M_5 of Marxen and Buntrock [Bulletin EATCS No. **40** (1990) 247–251] has $\Sigma(M_5) = 47,176,870$. Michel shows that M_5 halts on all inputs if and only if iterating the function $g(3m) = 5m + 6$, $g(3m + 1) = 5m + 9$, $g(3m + 2) = \uparrow$ eventually halts at \uparrow for all inputs. Similar results are proved for several other 5-state Turing machines. The result of Mahler [J. Aust. Math. Soc. **8** (1968) 313–321] on Z -numbers is restated in this framework: If a Z -number exists then there is an input m_0 such that the iteration $g(2m) = 3m$, $g(4m + 1) = 6m + 2$, $g(4m + 3) = \uparrow$ never halts with \uparrow when started from m_0 .

114. Filippo Mignosi (1995), *On a Generalization of the $3x + 1$ Problem*, J Number Theory, **55** (1995), 28–45. (MR 96m:11016).

For real $\beta > 1$, define the function $T_\beta : \mathbb{N} \rightarrow \mathbb{N}$ by

$$T_\beta(n) = \begin{cases} \lfloor \beta n \rfloor & \text{if } n \equiv 1 \pmod{2} \\ \frac{n}{2} & \text{if } n \equiv 0 \pmod{2} . \end{cases}$$

Then $\beta = \frac{3}{2}$ gives the $(3x + 1)$ -function. Conjecture C_β asserts that T_β has finitely many cycles and every $n \in \mathbb{N}$ eventually enters a cycle under iteration of T_β . The author shows that, for any fixed $0 < \epsilon < 1$, and for β either transcendental or rational with an even denominator, if $1 < \beta < 2$, then the set $S(\epsilon, \beta) = \{n : \text{some } T_\beta^{(k)}(n) < \epsilon n\}$ has natural density one, while if $\beta > 2$ then $S^*(\epsilon, \beta) = \{n : \text{some } T_\beta^{(k)}(n) > \epsilon^{-1}n\}$ has natural density one. For certain algebraic β different behavior may occur, and Conjecture C_β can sometimes be settled. In particular Conjecture C_β is true for $\beta = \sqrt{2}$ and false for $\beta = \frac{1+\sqrt{5}}{2}$.

115. Herbert Möller (1978), *Über Hasses Verallgemeinerung des Syracuse-Algorithmus (Kakutani Problem)*, Acta Arith. **34** (1978), No. 3, 219–226. (MR 57 #16246).

This paper studies for parameters d, m with $d \geq 2, m \geq 1$ and $(m, d) = 1$ the class of maps $H : \mathbb{Z}^+ \setminus \mathbb{Z}^+ \rightarrow \mathbb{Z}^+ \setminus d\mathbb{Z}^+$ having the the form

$$H(x) = \frac{1}{d^{a(x)}}(mx - r_j) \quad \text{when} \quad rx \equiv j \pmod{m}, \quad 1 \leq j \leq d-1,$$

in which $R(d) := \{r_j : 1 \leq j \leq d-1\}$ is any set of integers satisfying $r_j \equiv mj \pmod{d}$ for $1 \leq j \leq d-1$, and $d^{a(x)}$ is the maximum power of d dividing $mx - r_j$. The $3x + 1$ problem corresponds to $d = 2, m = 3$ and $R(d) = \{-1\}$. The paper shows that if

$$m \leq d^{d/(d-1)}$$

then the set of positive integers n which have some iterate $H^{(k)}(n) < n$, has full natural density $1 - \frac{1}{d}$ in the set $\mathbb{Z}^+ \setminus d\mathbb{Z}^+$. He conjectures that when $m \leq d^{d/(d-1)}$ the exceptional set of positive integers which don't satisfy the condition is finite.

The author's result generalizes that of Terras(1976) and Everett (1977). In a note added in proof the author asserts that the proofs of Terras (1976) are faulty. Terras's proofs seem essentially correct to me, and in response Terras (1979) provided further details.

116. Helmut A. Müller (1991), *Das '3n + 1' Problem*, Mitteilungen der Math. Ges. Hamburg **12** (1991), 231–251. (MR 93c:11053).

This paper presents basic results on $3x + 1$ problem, with some overlap of Lagarias (1985), and it presents complete proofs of the results it states. It contains new observations on the $3x + 1$ function T viewed as acting on the 2-adic integers \mathbb{Z}_2 . For $\alpha \in \mathbb{Z}_2$ the 2-adic valuation $|\alpha|_2 = 2^{-j}$ if $2^j \parallel \alpha$. Müller observes that $T(\alpha) = \sum_{n=0}^{\infty} (-1)^{n-1} (n+1) 2^{n-2} \binom{\alpha}{n}$ where $\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$. The function $T(\alpha)$ is locally constant but not analytic. Define the function $Q_\infty : \mathbb{Z}_2 \rightarrow \mathbb{Z}_2$ by $Q_\infty(\alpha) = \sum_{i=0}^{\infty} a_i 2^i$ where $a_i = 0$ if $|T^{(i)}(\alpha)|_2 < 1$ and $a_i = 1$ if $|T^{(i)}(\alpha)|_2 = 1$. Lagarias (1985) showed that this function is a measure-preserving homeomorphism of \mathbb{Z}_2 to itself. Müller proves that Q_∞ is nowhere differentiable.

117. Helmut A. Müller (1994), *Über eine Klasse 2-adischer Funktionen im Zusammenhang mit dem "3x + 1"-Problem*, Abh. Math. Sem. Univ. Hamburg **64** (1994), 293–302. (MR 95e:11032).

Let $\alpha = \sum_{j=0}^{\infty} a_{0,j} 2^j$ be a 2-adic integer, and let $T^{(k)}(\alpha) = \sum_{j=0}^{\infty} a_{k,j} 2^j$ denote its k -th iterate. Müller studies the functions $Q_j : \mathbb{Z}_2 \rightarrow \mathbb{Z}_2$ defined by $Q_j(\alpha) = \sum_{k=0}^{\infty} a_{k,j} 2^k$. He proves that each function Q_j is continuous and nowhere differentiable. He proves that the function $f : \mathbb{Z}_2 \rightarrow \mathbb{Q}_2$ given by $f = \sum_{j=1}^N A_j Q_j(\alpha)$ with constants $A_j \in \mathbb{Q}_2$ is differentiable at a point α with $T^{(k)}(\alpha) = 0$ for some $k \geq 0$ if and only if $2A_0 + A_1 = 0$ and $A_2 = A_3 = \dots = A_N = 0$.

118. Władysław Narkiewicz (1980), *A note on a paper of H. Gupta concerning powers of 2 and 3*, Univ. Beograd. Publ. Elektrotech. Fak. Ser. Mat. Fiz. No **678-715** (1980), 173–174.

Erdős raised the question: "Does there exist an integer $m \neq 0, 2, 8$ such that 2^m is a sum of distinct integral powers of 3?" This was motivated by work of Gupta [Univ. Beograd. Publ. Elektrotech. Fak. Ser. Mat. Fiz. Nos. **602-633** (1980), 151–158]. who checked numerically that the only solutions were $m = 0, 2$ and 8 , for $m \leq 4734$. This paper shows that if $N(T)$ denotes the number of such $m \leq T$ then $N(T) \leq 1.62T^\theta$ where $\theta = \frac{\log 2}{\log 3}$.

119. C. Stanley Ogilvy (1972), *Tomorrow's Math: unsolved problems for the amateur*, Second Edition, Oxford University Press: New York 1972.

The $3x + 1$ problem is discussed on pages 103–104. He states: "H. S. M. Coxeter, who proposed it in 1970, stated then that it had been checked for all $N \leq 500,000$. However if the conjecture is true, which seems likely, a proof will have nothing to do with computers." He notes that the analogous conjecture for the $5x + 1$ problem is false, since there is a cycle not reaching 1.

Note. See Coxeter (1971), where Coxeter reports the problem "as a piece of mathematical gossip."

120. Tomás Oliveira e Silva (1999), *Maximum Excursion and Stopping Time Record-Holders for the 3x + 1 Problem: Computational Results*, Math. Comp. **68** No. 1 (1999), 371–384, (MR 2000g:11015).

This paper reports on computations that verify the $3x+1$ conjecture for $n < 3 \cdot 2^{53} = 2.702 \times 10^{16}$. It also reports the values of n that are champions for the quantity $\frac{t(n)}{n}$, where

$$t(n) := \sup_{k \geq 1} T^{(k)}(n) .$$

In this range all $t(n) \leq 8n^2$, which is consistent with the conjecture $t(n) \leq n^{2+o(1)}$ of Lagarias and Weiss (1992).

Note: In 2004 he implemented an improved version of this algorithm. As of 2008 his computation verified the $3x+1$ conjecture up to $19 \times 2^{58} \approx 5.476 \times 10^{18}$. See his webpage at: <http://www.ieeta.pt/~tos/>; email: tos@ieeta.pt. This is the current record value for verifying the $3x+1$ conjecture.

121. Elio Oliveri and Giuseppe Vella (1998), *Alcune Questioni Correlate al Problema Del “ $3D+1$ ”*, Atti della Accademia di scienze lettere e arti di Palermo, Ser. V, **28** (1997-98), 21–52. (Italian)

This paper studies the structure of forward and backward iterates of the $3x+1$ problem, treating the iteration as proceeding from one odd integer to the next odd iterate. It obtains necessary conditions for existence of a nontrivial cycle in the $3x+1$ problem. It observes that if (D_1, \dots, D_n) are the odd integers in such a cycle, then

$$\prod_{i=1}^n \frac{3D_i + 1}{D_i} = 2^{e_1 + \dots + e_n} = 2^k .$$

If the $3x+1$ conjecture is verified up to D then $\frac{3D+1}{D} > \frac{3D_i+1}{D_i}$, for all i , whence

$$n \log_2 \left(\frac{3D+1}{2} \right) > k > n \log_2 3 .$$

They conclude that there must be an integer in the interval $[n \log_2(\frac{3D+1}{2}), n \log_2 3]$, a condition which imposes a lower bound on n , the number of odd integers in the cycle. In fact one must have $k = 1 + \lfloor n \log_2 3 \rfloor$, and $\text{Max}_i \{D_i\} > (2^{\frac{k}{n}} - 3)^{-1}$, and $\frac{k}{n}$ is a good rational approximation to $\log_2 3$. Using the continued fraction expansion of $\log_2 3$, including the intermediate convergents, the authors conclude that if the $3x+1$ conjecture is verified for all integers up to $D = 2^{40} + 1$, then one may conclude $n \geq 1078215$.

The paper also investigates the numbers $M(N, n)$ of parity-sequences of length N which contain n odd values, at positions k_1, k_2, \dots, k_n with $k_n = N$, and such that the partial sums $h_j = k_1 + k_2 + \dots + k_j$ with all $h_j \leq L_j := \lfloor j \log_2 3 \rfloor$. Here $\log_2(3) \approx 1.585$. These are the possible candidate initial parity sequences for the smallest positive number in a nontrivial cycle of the $3x+1$ iteration of period N or longer.

Note: The method of the authors for getting a lower bound on the length of non-trivial cycles is similar in spirit to earlier methods, cf. Eliahou (1993) and Halbeisen and H ungerb uhler (1997). Later computations of the authors show that if one can take $D = 2^{61} + 1 \approx 2.306 \times 10^{18}$, then one may conclude there are no nontrivial cycles containing less than $n = 6,586,818,670$ odd integers. The calculations of T. Olivera e Silva, as of 2008, have verified the $3x+1$ conjecture to $19 \times 2^{58} \approx 5.476 \times 10^{18}$, so this improved cycle bound result is now unconditional.

122. Clifford A. Pickover (1989), *Hailstone $3n + 1$ Number graphs*, J. Recreational Math. **21** (1989), 120–123.

This paper gives two-dimensional graphical plots of $3x + 1$ function iterates revealing “several patterns and a diffuse background of chaotically-positioned dots.”

123. Margherita Pierantoni and Vladan Ćurčić (1996), *A transformation of iterated Collatz mappings*, Z. Angew Math. Mech. **76**, Suppl. 2 (1996), 641–642. (Zbl. 900.65373).

A generalized Collatz map has the form $T(x) = a_i x + b_i$, for $x \equiv i \pmod{n}$, in which $a_i = \frac{\alpha_i}{n}$, $b_i = \frac{\beta_i}{n}$, with α_i, β_i integers satisfying $i\alpha_i + \beta_i \equiv 0 \pmod{n}$. The authors note there is a continuous extension of this map to the real line given by

$$\hat{T}(x) = \sum_{m=0}^{n-1} (a_m x + b_m) \left(\sum_{h=0}^{n-1} e^{\frac{2\pi i h(m-x)}{n}} \right) = \sum_{h=0}^{n-1} (A_h x + B_h) e^{-\frac{2\pi i h x}{n}},$$

where $\{A_k\}, \{B_k\}$ are the discrete Fourier transforms of the $\{a_i\}$, res. $\{b_i\}$. This permits analysis of the iterations of the map using the discrete Fourier transform and its inverse.

They specialize to the case $n = 2$, where the data (a_0, a_1, b_0, b_1) describing T are half-integers, with b_0 and $a_1 + b_1$ integers. The $3x + 1$ function corresponds to $(a_0, a_1, b_0, b_1) = (\frac{1}{2}, \frac{3}{2}, 0, \frac{1}{2})$. One has $A_0 = \frac{1}{2}(a_0 + a_1)$, $A_1 = \frac{1}{2}(a_0 - a_1)$, $B_0 = \frac{1}{2}(b_0 + b_1)$, $B_1 = \frac{1}{2}(b_0 - b_1)$. The authors note that the recursion

$$x_{k+1} = \hat{T}(x_k) = (A_0 x_k + B_0) + (A_1 x_k + B_1) \cos(\pi x_k)$$

can be transformed into a two-variable system using the auxiliary variable $\xi_k = \cos(\pi x_k)$, as

$$\begin{aligned} x_{k+1} &= (A_0 x_k + B_0) + (A_1 x_k + B_1) \xi_k \\ \xi_{k+1} &= \cos(\pi(A_0 x_k + B_0) + \pi(A_1 x_k + B_1) \xi_k). \end{aligned}$$

They give a formula for x_m in terms of the data $(x_0, \xi_0, \xi_1, \dots, \xi_{m-1})$, namely

$$x_m = \left(\prod_{j=0}^{m-1} (A_0 + A_1 \xi_j) \right) x_0 + (B_0 + \xi_{m-1} B_1) + \sum_{k=0}^{m-2} (B_0 + \xi_k B_1) \prod_{j=k+1}^{m-1} (A_0 + \xi_k A_1).$$

Now they study the transformed system in terms of the auxiliary variables ξ_j . When the starting values (x_0, x_1) are integers, all subsequent values are integers, and then the auxiliary variables $\xi_k = \pm 1$. Then the recursion for ξ_{k+1} above can be simplified by trigonometric sum of angles formulas. In particular they obtain recursions for the ξ_j that are independent of the x'_k s whenever a_0 and a_1 are both integers, but not otherwise. Finally the authors observe that on integer orbits ξ_m is a periodic function of x_0 (of period dividing 2^{m+1}) which can be interpolated using a Fourier series in $\cos \frac{2\pi h}{2^{m+1}}$, $\sin \frac{2\pi h}{2^{m+1}}$, using the inverse discrete Fourier transform. They give explicit interpolations for ξ_m for the $3x + 1$ function for $m = 0, 1, 2$.

124. Susana Puddu (1986), *The Syracuse problem (Spanish)*, 5th Latin American Colloq. on Algebra – Santiago 1985, Notas Soc. Math. Chile **5** (1986), 199–200. (MR88c:11010).

This note considers iterates of the Collatz function $C(x)$. It shows every positive m has some iterate $C^k(m) \equiv 1 \pmod{4}$. If $m \equiv 3 \pmod{4}$ the smallest such k must have $C^k(m) \equiv 5 \pmod{12}$.

125. Qiu, Wei Xing (1997), *Study on “ $3x+1$ ” problem* (Chinese), J. Shanghai Univ. Nat. Sci. Ed. [Shanghai da xue xue bao. Zi ran ke xue ban] **3** (1997), No. 4, 462–464.

English Abstract: “The paper analyses the structure presented in the problem “ $3x+1$ ” and points out there is no cycle in the problem except that $x = 1$.”

126. Raymond Queneau (1963), *Note complémentaire sur la Sextine*, Subsidia Pataphysica, No.1 (1963), 79–80.

Raymond Queneau (1903-1976) was a French poet and novelist, and a founding member in 1960 of the French mathematical-literary group Oulipo (Ouvroir de littérature potentielle). He was a member of the mathematical society of France from 1948 on, and published in 1972 a mathematical paper in additive number theory. His final essay in 1976 was titled: “Les fondaments de la littérature d’après David Hilbert” (La Bibliothèque Oulipienne, No. 3); it set out an axiomatic foundation of literature in imitation of Hilbert’s *Foundations of Geometry*, replacing “points”, “straight line”, and “plane” with “word”, “sentence” and “paragraph”, respectively, in some of Hilbert’s axioms. [English translation: The Foundations of Literature (after David Hilbert), in: R. Queneau, I. Calvino, P. Fournel, J. Jouet, C. Berge and H. Mathews, *Oulipo Laboratory, Texts from the Bibliothèque Oulipienne*, Atlas Press: Bath 1995.] He deduces from his axioms: “THEOREM 7. *Between two words of a sentence there exists an infinity of other words.*” To explain this result, he posits the existence of “imaginary words.”

This short note is a comment on a preceding article by A. Taverna, *Arnaut Daniel et la Spirale*, Subsidia Pataphysica, No. 1 (1969), 73–78. Arnaut Daniel was a 12-th century troubadour who composed poems in Occitan having a particular rhyming pattern, called a sestina. Dante admired him and honored him in several works, including the *Divine Comedy*, where Daniel is depicted as doing penance in Purgatory in *Purgatorio*. The rhyming pattern of the sestina had six sextets with rhyme pattern involving a cyclic permutation of order 6, followed by a triplet. The work of Taverna observes that this cyclic permutation can be represented using a spiral pattern.

Queneau considers the “spiral permutation” on numbers $\{1, 2, \dots, n\}$ which takes $2p$ to p and $2p+1$ to $n-p$. He raises the question: For which n is a similar spiral permutations in the symmetric group S_n a cyclic permutation? Call these allowable n . The example of the sestina is the case $n = 6$; this pattern he terms the sextine. He says it is easy to show that numbers of the form $n = 2xy + x + y$, with $x, y \geq 1$, are not allowable numbers; this excludes $n = 4, 7, 10$ etc. He states that the following 31 integers $n \leq 100$ are allowable: $n = 1, 2, 3, 5, 6, 9, 11, 14, 18, 23, 26, 29, 30, 33, 35, 39, 41, 50, 51, 53, 65, 69, 74, 81, 83, 86, 89, 90, 95, 98, 99$.

This paper is in the Oulipo spirit, considering literature obtainable when mathematical restrictions are placed on its allowable form. Queneau also discussed the topic of “Sextines” in his 1965 essay on the aims of the group Oulipo, “Littérature potentielle”, published in *Bâtons, chiffres et lettres*, 2nd Edition, Gallimard: Paris 1965. [English translation: Potential Literature, pp. 181–196 in R. Queneau, *Letters, Numbers, Forms: Essays 1928–1970* (Jordan Stump, Translator), Univ. of Illinois Press: Urbana and Chicago 2007.]

Note. Queneau’s question was later formulated as the behavior under iteration of a $(3x + 1)$ -like function $\delta_n(x)$ on the range $x \in \{1, 2, \dots, n\}$, as observed in Roubaud (1969) and Bringer (1969). Queneau later published a paper in additive number theory [J. Comb. Theory A **12** (1972), 31–71], which is on a different topic, but mentions in passing at least one $(3x + 1)$ -like function (on page 63).

127. Lee Ratzan (1973), *Some work on an Unsolved Palindromic Algorithm*, Pi Mu Epsilon Journal **5** (Fall 1973), 463–466.

The $3x + 1$ problem and a generalization were proposed as an Undergraduate Research Project by David Kay (1972). The author gives computer code to test the conjecture, and used it to verify the $3x + 1$ Conjecture up to 31,910. He notices some patterns in two consecutive numbers having the same total stopping time and makes conjectures when they occur.

128. Daniel A. Rawsthorne (1985), *Imitation of an iteration*, Math. Magazine **58** (1985), 172–176. (MR 86i:40001).

This paper proposes a multiplicative random walk that models imitating the “average” behavior of the $3x + 1$ function and similar functions. It compares the mean and standard deviation that this model predicts with empirical $3x + 1$ function data, and with data for several similar mappings, and finds good agreement between them.

129. H. J. J. te Riele (1983a), *Problem 669*, Nieuw Archief voor Wiskunde, Series IV, **1** (1983), p. 80.

This problem asks about the iteration of the function

$$f(x) = \begin{cases} \frac{n}{3} & \text{if } n \equiv 0 \pmod{3} \\ \lfloor n\sqrt{3} \rfloor & \text{if } n \not\equiv 0 \pmod{3} . \end{cases}$$

on the positive integers. The value $n = 1$ is a fixed point, and all powers $n = 3^k$ eventually reach this fixed point. The problem asks to show that if two consecutive iterates are not congruent to 0 (mod 3) then the trajectory of this orbit thereafter grows monotonically, so diverges to $+\infty$; otherwise the orbit reaches 1.

Note. Functions similar in form to $f(x)$, but with a condition (mod 2), were studied by Mignosi (1995) and Brocco (1995).

130. H. J. J. te Riele (1983b), *Iteration of Number-theoretic functions*, Nieuw Archief voor Wiskunde, Series IV, **1** (1983), 345–360. [MR 85e:11003].

The author surveys work on a wide variety of number-theoretic functions which take positive integers to positive integers, whose behavior under iteration is not understood. This includes the $3x + 1$ function (III.1) and the $qx + 1$ function (III.2). The author presents as Example 1 the function in te Riele (1983a),

$$f(x) = \begin{cases} \frac{n}{3} & \text{if } n \equiv 0 \pmod{3} \\ \lfloor n\sqrt{3} \rfloor & \text{if } n \not\equiv 0 \pmod{3} . \end{cases}$$

He showed that this function has divergent trajectories, and that all non-divergent orbits converge to the fixed point 1. He conjectures that almost all orbits on positive integers tend to $+\infty$. Experimentally only 459 values of $n < 10^5$ have orbits converging to 1.

131. Jacques Roubaud (1969) , *Un problème combinatoire posé par la poésie lyrique des troubadours*, Mathématiques et Sciences Humaines [Mathematics and Social Science], **27** Autumn 1969, 5–12.

This paper addresses the question of suitable rhyming schemes for poems, suggested by the schemes used by medieval troubadours, and raised in Queneau (1963). This leads to combinatorial questions concerning the permutation structure of such rhyming schemes. The author classifies the movement of rhymes by permutation patterns. In section 5 he formulates three questions suggested by the rhyme pattern of the sestina of Arnaut Daniel. One of these (problem (Pa)) is the question of Queneau (1963), which he states as iteration of the $3x + 1$ -like function

$$\delta_n(x) := \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ \frac{2n+1-x}{2} & \text{if } x \text{ is odd} \end{cases}$$

restricted to the domain $\{1, 2, \dots, n\}$. This paper goes together with the analysis of this function done by his student Monique Bringer (1969). In Robaud (1993) he proposes further generalizations of these problems.

Note. Jacques Roubaud is a mathematician and a member of the literary group Oulipo. In 1986 he wrote a spoof, an obituary for N. Bourbaki (see pages 73 and 115 in: M. Mashaal, *Bourbaki* (A. Pierrehumbert, Trans.), Amer. Math. Soc., Providence 2006). He discussed the mathematical work of Raymond Queneau in J. Roubaud, *La mathématique dans la méthode de Raymond Queneau*, Critique: revue générale des publications françaises et étrangères **33** (1977), no. 359, 392–413. [English Translation: Mathematics in the Method of Raymond Queneau, pp. 79–96 in: Warren F. Motte, Jr. (Editor and Translator), *Oulipo, A Primer of Potential Literature*, Univ. of Nebraska Press: Lincoln, NB 1986.]

132. Jacques Roubaud (1993), *N-ine, autrement dit quenine (encore)*, *Réflexions historiques et combinatoires sur la n-ine, autrement dit quenine*. La Bibliothèque Oulipienne, numéro 66, Rotographie à Montreuil (Seine-Saint-Denis), November 1993.

This paper considers work on rhyming patterns in poems suggested by those of medieval troubadours. The n -ine is a generalization of the rhyme pattern of the sestina of Arnaut Daniel, a 12-th century troubadour, see Queneau (1963), Roubaud (1969). It considers a “spiral permutation” of the symmetric group S_N , which he earlier gave mathematically as

$$\delta_n(x) := \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ \frac{2n+1-x}{2} & \text{if } x \text{ is odd} \end{cases}$$

restricted to the domain $\{1, 2, \dots, n\}$.

Roubaud summarizes the work of Bringer (1969) giving restrictions on admissible n . She observed $p = 2n+1$ must be prime and that 2 is a primitive root of p is a sufficient condition

for admissibility. He advances the conjectures that the complete set of admissible n are those with $p = 2n + 1$ a prime, such that either $\text{ord}_p(2) = 2n$, so that 2 is a primitive root, or else $\text{ord}_p(2) = n$. (This conjecture later turned out to require a correction; namely, to allow those n with $\text{ord}_p(2) = n$ only when $n \equiv 3 \pmod{4}$.)

Roubaud suggests various generalizations of the problem. Bringer(1969) had studied the inverse permutation to δ_n , given by

$$d_n(x) := \begin{cases} 2x & \text{if } 1 \leq x \leq \frac{n}{2} \\ 2n + 1 - 2x & \text{if } \frac{n}{2} < x \leq n \end{cases}$$

In Section 5 Roubaud suggests a generalization of this, to the permutations

$$d_n(x) := \begin{cases} 3x & \text{if } 1 \leq x \leq \frac{n}{3} \\ 2n + 1 - 3x & \text{if } \frac{n}{3} < x \leq \frac{2n}{3} \\ 3x - (2n + 1) & \text{if } \frac{2n}{3} < x \leq n. \end{cases}$$

Here $n = 8$ is a solution, and he composes a poem, “Novembre”, a 3-octine, to this rhyming pattern, in honor of Raymond Queneau. Call these solutions 3-admissible, and those of the original problem 2-admissible. He gives the following list of 3-admissible solutions for $n \leq 200$ that are not 2-admissible: $n = 8, 15, 21, 39, 44, 56, 63, 68, 111, 116, 128, 140, 165, 176, 200$.

He concludes with some other proposed rhyming schemes involving spirals, including the “spinine”, also called “escargonine”.

133. Olivier Rozier (1990), *Demonstraton de l'absence de cycles d'une certain forme pour le problème de Syracuse*, Singularité **1** no. 3 (1990), 9–12.

This paper proves that there are no cycles except the trivial cycle whose iterates (mod 2) repeat a pattern of the form $1^m 0^{m'}$. Such cycles are called *circuits* by R. P. Steiner, who obtained this result in 1978. Rozier’s proof uses effective transcendence bounds similar to Steiner’s, but is simpler since he can quote the recent bound

$$\left| \frac{\log 3}{\log 2} - \frac{p}{q} \right| > q^{-15},$$

when $(p, q) = 1$, which appears in M. Waldschmidt, *Equationes diophantiennes et Nombres Transcendents*, Revue Palais de la Découverte, **17**, No. 144 (1987) 10–24.

134. Olivier Rozier (1991), *Probleme de Syracuse: “Majorations” Elementaires des Cycles*, Singularité **2**, no. 5 (1991), 8–11.

This paper proves that if the integers in a cycle of the $3x + 1$ function are grouped into blocks of integers all having the same parity, for any k there are only a finite number of cycles having $\leq k$ blocks. The proof uses the Waldschmidt result $\left| \frac{\log 3}{\log 2} - \frac{p}{q} \right| > q^{-15}$.

135. Jürgen W. Sander (1990), *On the $(3N + 1)$ -conjecture*, Acta Arithmetica **55** (1990), 241–248. (MR 91m:11052).

The paper shows that the number of integers $\leq x$ for which the $3x + 1$ Conjecture is true is at least $x^{3/10}$, by extending the approach of Crandall (1978). (Krasikov (1989) obtains a better lower bound $x^{3/7}$ by another method.) This paper also shows that if the $3x + 1$ Conjecture is true for all $n \equiv \frac{1}{3}(2^{2k} - 1) \pmod{2^{2k-1}}$, for any fixed k , then it is true in general.

136. Benedict G. Seifert (1988), *On the arithmetic of cycles for the Collatz-Hasse ('Syracuse') conjectures*, Discrete Math. **68** (1988), 293–298. (MR 89a:11031).

This paper gives criteria for cycles of $3x + 1$ function to exist, and bounds the smallest number in the cycle in terms of the length of the cycle. Shows that if the $3x + 1$ Conjecture is true, then the only positive integral solution of $2^l - 3^r = 1$ is $l = 2, r = 1$.

[All integer solutions of $2^l - 3^r = 1$ have been found unconditionally. This can be done by a method of Størmer, Nyt. Tidsskr. Math. B **19** (1908) 1–7. It was done as a special case of various more general results by S. Pillai, Bull. Calcutta Math. Soc. **37** (1945), 15–20 (MR# 7, 145i); W. LeVeque, Amer. J. Math. **74** (1952), 325–331 (MR 13, 822f); R. Hampel, Ann. Polon. Math. **3** (1956), 1–4 (MR 18, 561c); etc.]

137. Jeffrey O. Shallit and David W. Wilson (1991), *The “ $3x + 1$ ” Problem and Finite Automata*, Bulletin of the EATCS (European Association for Theoretical Computer Science), No. 46, 1991, pp. 182–185.

A set S of positive integers is said to be *2-automatic* if the binary representations of the integers in S form a regular language $L_S \subseteq \{0, 1\}^*$. Let S_i denote the set of integers n which have some $3x + 1$ function iterate $T^{(j)}(n) = 1$, and whose $3x + 1$ function iterates include exactly i odd integers ≥ 3 . The sets S_i are proved to be 2-automatic for all $i \geq 1$.

138. Daniel Shanks (1965), *Comments on Problem 63 – 13**, SIAM Review **7** (1965), 284–286.

This note gives comments on Problem 63 – 13*, proposed by Klamkin (1963). This problem concerns the iteration of Collatz’s original function, which is a permutation of the integers. He states that these problems date back at least to 1950, when L. Collatz mentioned them in personal conversations at the International Math. Congress held at Harvard University. He gives the results of a computer search of the orbit of 8 for Collatz’s original function, observing that it reaches numbers larger than 10^{10} . He observes there are known cycles of length 1, 2, 5 and 12, the last having smallest element $n = 144$. He observes that this seems related to the fact that the continued fraction expansion of $\log_2 3$ has initial convergents having denominators 1, 2, 5, 12, 41, ... However he does not know of any cycle of period 41. He notes that it is not known whether the only cycle lengths that can occur must be denominators of such partial quotients. Later Atkin (1966) proved there exists no cycle of period 41.

139. Daniel Shanks (1975), *Problem #5*, Western Number Theory Conference 1975, Problem List, (R. K. Guy, Ed.).

The problem concerns iteration of the Collatz function $C(n)$. Let $l(n)$ count the number of distinct integers appearing in the sequence of iterates $C^{(k)}(n)$, $k \geq 1$, assuming it eventually enters a cycle. Thus $l(1) = 3, l(2) = 3, l(3) = 8$, for example. Set

$S(N) = \sum_{n=1}^N l(n)$. The problem asks whether it is true that

$$S(N) = AN \log N + BN + o(N) \quad \text{as } N \rightarrow \infty,$$

where

$$A = \frac{3}{2} \log \frac{4}{3} \approx 5.21409 \quad \text{and} \quad B = A(1 - \log 2) \approx 1.59996.$$

In order for $S(N)$ to remain finite, there must be no divergent trajectories. This problem formalizes the result of a heuristic probabilistic calculation based on assuming the $3x + 1$ Conjecture to be true.

140. Ray P. Steiner (1978), *A Theorem on the Syracuse Problem*, Proc. 7-th Manitoba Conference on Numerical Mathematics and Computing (Univ. Manitoba-Winnipeg 1977), Congressus Numerantium XX, Utilitas Math.: Winnipeg, Manitoba 1978, pp. 553–559. (MR 80g:10003).

This paper studies periodic orbits of the $3x + 1$ map, and a problem raised by Davidson (1976). A sequence of iterates $\{n_1, n_2, \dots, n_p, n_{p+1}\}$ with $T(n_j) = n_{j+1}$ is called by Davidson (1977) a *circuit* if it consists of a sequence of odd integers $\{n_1, n_2, \dots, n_j\}$ followed by a sequence of even integers $\{n_{j+1}, n_{j+2}, \dots, n_p\}$, with $n_{p+1} = T(n_p)$ an odd integer. A circuit is a *cycle* if $n_{p+1} = n_1$.

This paper shows that the only circuit on the positive integers that is a cycle is $\{1, 2\}$. It uses the observation of Davison (1977) that these corresponds to positive solutions (k, l, h) to the exponential Diophantine equation

$$(2^{k+l} - 3^k)h = 2^l - 1.$$

The paper shows that the only solution to this equation in positive integers is $(k, l, h) = (1, 1, 1)$. The proof uses results from transcendence theory, Baker's method of linear forms in logarithms (see A. Baker, *Transcendental Number Theory*, Cambridge Univ. Press 1975, p. 45.)

[One could prove similarly that there are exactly three circuits that are cycles on the non-positive integers, namely $\{-1\}$, and $\{-5, -7, -10\}$. These correspond to the solutions to the exponential Diophantine equation $(k, l, h) = (k, 0, 0)$ for any $k \geq 1$; and $(2, 1, -1)$, respectively. A further solution $(0, 2, 1)$ corresponds to the cycle $\{0\}$ which is not a circuit by the definition above.]

141. Ray P. Steiner (1981a), *On the “ $Qx + 1$ ” Problem, Q odd*, Fibonacci Quarterly **19** (1981), 285–288. (MR 84m:10007a)

This paper studies the $Qx + 1$ -map

$$h(n) = \begin{cases} \frac{Qn+1}{2} & \text{if } n \equiv 1 \pmod{2}, n > 1 \\ \frac{n}{2} & \text{if } n \equiv 0 \pmod{2} \\ 1 & \text{if } n = 1, \end{cases}$$

when Q is odd and $Q > 3$. It proves that the only circuit which is a cycle when $Q = 5$ is $\{13, 208\}$, that there is no circuit which is a cycle for $Q = 7$. Baker's method is again used, as in Steiner (1978).

142. Ray P. Steiner (1981b), *On the “ $Qx + 1$ ” Problem, Q odd II*, Fibonacci Quarterly **19** (1981), 293–296. (MR 84m:10007b).

This paper continues to study the $Qx + 1$ -map

$$h(n) = \begin{cases} \frac{Qn+1}{2} & \text{if } n \equiv 1 \pmod{2}, n > 1 \\ \frac{n}{2} & \text{if } n \equiv 0 \pmod{2} \\ 1 & \text{if } n = 1, \end{cases}$$

when Q is odd and $Q > 3$. It makes general remarks on the case $Q > 7$, and presents data on from the computation of $\log_2 \frac{5}{2}$ and $\log_2 \frac{7}{2}$ used in the proofs in part I.

143. Rosemarie M. Stemmler (1964), *The ideal Waring problem for exponents* 401 – 200,000, Math. Comp. **18** (1964), 144–146. (MR 28 #3019)

The ideal Waring theorem states that for a given $k \geq 2$ each positive integer is the sum of $2^k + \lfloor (\frac{3}{2})^k \rfloor$ non-negative k -th powers, provided that the fractional part $\{\{x\}\} := x - \lfloor x \rfloor$ of $(\frac{3}{2})^k$ satisfies

$$0 \leq \{\{(\frac{3}{2})^k\}\} < 1 - (\frac{3}{4})^k.$$

This paper checks that this inequality holds for $401 \leq k \leq 200000$. This problem on the fractional parts of powers of $(\frac{3}{2})^k$ motivated the work of Mahler (1968) on Z -numbers.

144. Kenneth S. Stolarsky (1998), *A prelude to the $3x + 1$ problem*, J. Difference Equations and Applications **4** (1998), 451–461. (MR 99k:11037).

This paper studies a purportedly “simpler” analogue of the $3x + 1$ function. Let $\phi = \frac{1+\sqrt{5}}{2}$. The function $f : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ is given by

$$\begin{cases} f(\lfloor n\phi \rfloor) = \lfloor n\phi^2 \rfloor + 2, & \text{all } n \geq 1, \\ f(\lfloor n\phi^2 \rfloor) = n. & \text{all } n \geq 1. \end{cases}$$

This function is well-defined because the sets $A = \{\lfloor n\phi \rfloor : n \geq 1\}$ and $B = \{\lfloor n\phi^2 \rfloor : n \geq 1\}$ form a partition of the positive integers. (This is a special case of Beatty’s theorem.) The function f is analogous to the $3x + 1$ function in that it is increasing on the domain A and decreasing on the domain B . The paper shows that almost all trajectories diverge to $+\infty$, the only exceptions being a certain set $\{3, 7, 18, 47, \dots\}$ of density zero which converges to the two-cycle $\{3, 7\}$. Stolarsky determines the complete symbolic dynamics of f . The possible symbol sequences of orbits are $B^\ell(AB)^kA^\infty$ for some $\ell, k \geq 0$, for divergent trajectories, and $B^\ell(AB)^\infty$ for some $\ell \geq 0$, for trajectories that reach the two-cycle $\{3, 7\}$.

145. György Targónski (1991), *Open questions about KW-orbits and iterative roots*, Aequationes Math. **41** (1991), 277–278.

The author suggests the possibility of applying a result of H. Engl (An analytic representation for self-maps of a countably infinite set and its cycles, Aequationes Math.

25 (1982), 90-96, MR85d.04001) to bound the number of cycles of the $3x + 1$ problem. Engl's result expresses the number of cycles as the geometric multiplicity of 1 as an eigenvalue of a map on the sequence space l^1 .

146. Riho Terras (1976), *A stopping time problem on the positive integers*, Acta Arithmetica **30** (1976), 241–252. (MR 58 #27879).

This is the first significant research paper to appear that deals directly with the $3x + 1$ function. The $3x + 1$ function was however the motivation for the paper Conway (1972). The main result of this paper was obtained independently and contemporaneously by Everett (1977).

A positive integer n is said to have *stopping time* k if the k -th iterate $T^{(k)}(n) < n$, and $T^{(j)}(n) \geq n$ for $1 \leq j < k$. The author shows that the set of integers having stopping time k forms a set of congruence classes (mod 2^k), minus a finite number of elements. He shows that the set of integers having a finite stopping time has natural density one. Some further details of this proof were supplied later in Terras (1979).

This paper introduces the notion of the *coefficient stopping time* $\kappa(n)$ of an integer $n > 1$. Write $T^{(k)}(n) = \alpha(n)n + \beta(n)$ with $\alpha(n) = \frac{3^{a(n)}}{2^n}$, where $a(n)$ is the number of iterates $T^{(j)}(n) \equiv 1 \pmod{2}$ with $0 \leq j < k$. Then $\kappa(n)$ is defined to be the least $k \geq 1$ such that $T^{(k)}(n) = \alpha(n)n + \beta(n)$ has $\alpha(n) < 1$, and $\kappa(n) = \infty$ if no such value exists. It is clear that $\kappa(n) \leq \sigma(n)$, where $\sigma(n)$ is the stopping time of n . Terras formulates the *Coefficient Stopping Time Conjecture*, which asserts that $\kappa(n) = \sigma(n)$ for all $n \geq 2$. He proves this conjecture for all values $\kappa(n) \leq 2593$. This can be done for $\kappa(n)$ below a fixed bound by upper bounding $\beta(n)$ and showing bounding $\kappa(n) < 1 - \delta$ for suitable δ and determining the maximal value $\frac{3^l}{2^j} < 1$ possible with j below the given bound. The convergents of the continued fraction expansion of $\log_2 3$ play a role in determining the values of j that must be checked.

147. Riho Terras (1979), *On the existence of a density*, Acta Arithmetica **35** (1979), 101–102. (MR 80h:10066).

This paper supplies additional details concerning the proof in Terras (1976) that the set of integers having an infinite stopping time has asymptotic density zero. The proof in Terras (1976) had been criticized by Möller (1978).

148. Bryan Thwaites (1985), *My conjecture*, Bull. Inst. Math. Appl. **21** (1985), 35–41. (MR86j:11022).

The author states that he invented the $3x + 1$ problem in 1952. He derives basic results about iterates, and makes conjectures on the “average” behavior of trajectories.

Note. The mathematical community generally credits L. Collatz as being the first to propose the $(3x + 1)$ -problem, see Collatz (1986), and the comment of Shanks (1965). This would be an independent discovery of the problem.

149. Bryan Thwaites (1996), *Two Conjectures, or how to win £ 1000*, Math. Gazette **80** (1996), 35–36.

One of the two conjectures is the $3x + 1$ -problem, for which the author offers the stated reward.

150. Robert Tijdeman (1972), *Note on Mahler's 3/2-problem*, Det Kongelige Norske Videnskabs Selskab Skrifter No. 16, 1972, 4 pages. (Zbl. 227: 10025.)

This paper concerns the Z-number problem of Mahler (1968), which asks whether there exists any nonzero real number η such that the fractional parts $0 \leq \{\{\eta(\frac{3}{2})^n\}\} \leq \frac{1}{2}$ for all $n \geq 0$. By an elementary argument Tijdeman shows that analogues of Z-numbers exist in a related problem. Namely, for every $k \geq 2$ and $m \geq 1$ there exists a real number $\eta \in [m, m+1)$ such that $0 \leq \{\{\eta(\frac{2k+1}{2})^n\}\} \leq \frac{1}{2k+1}$ holds for all $n \geq 0$.

151. Charles W. Trigg, Clayton W. Dodge and Leroy F. Meyers (1976), *Comments on Problem 133*, Eureka (now Crux Mathematicorum) **2**, No. 7 (August-Sept.) (1976), 144–150.

Problem 133 is the $3x + 1$ problem. It was proposed by K. S. Williams (Concordia Univ.), who said that he was shown it by one of his students. C. W. Trigg gives some earlier history of the problem. He remarks that Richard K. Guy wrote to him stating that Lothar Collatz had given a lecture on the problem at Harvard in 1950 (informally at the International Math. Congress). He reported that in 1970 H. S. M. Coxeter offered a prize of \$50 for proving the $3x + 1$ Conjecture and \$100 for finding a counterexample, in his talk: “Cyclic Sequences and Frieze Patterns” (The Fourth Felix Behrend Memorial Lecture in Mathematics), The University of Melbourne, 1970, see Coxeter (1971). He also referenced a discussion of the problem in several issues of *Popular Computing* No. 1 (April 1973) 1–2; No. 4 (July 1973) 6–7; No. 13 (April 1974) 12–13; No. 25 (April 1975), 4–5. Dodge references the work of Isard and Zwicky (1970).

Note. Lothar Collatz was present at the 1950 ICM as part of the DMV delegation, while Coxeter, Kakutani and Ulam each delivered a lecture at the 1950 Congress that appears in its proceedings.

152. Toshio Urata and Katsufumi Suzuki (1996), *Collatz Problem*, (Japanese), Epsilon (The Bulletin of the Society of Mathematics Education of Aichi Univ. of Education) [Aichi Kyoiku Daigaku Sugaku Kyoiku Gakkai shi] **38** (1996), 123–131.

The authors consider iterating the Collatz function $C(x)$ (denoted $f(x)$) and the speeded-up Collatz function $\phi(x)$ mapping odd integers to odd integers by

$$\phi(x) = \frac{3x+1}{2^e}, \quad \text{with } 2^e || 3x+1.$$

They let i_n (resp. j_n) denote the number of iterations to get from n to 1 of the Collatz function (resp. the function $\phi(x)$). They study the Cesaro means

$$h_n := \sum_{k=1}^n i_k, \quad m_n := \sum_{k=1}^n j_k.$$

and observe empirically that

$$h_n \approx 10.4137 \log n - 12.56$$

$$m_n \approx 2.406 \frac{\log n}{\log 2} - 2$$

fit the data up to 4×10^8 . Then they introduce an entire function which interpolate the Collatz function at positive integer values

$$f(z) = \frac{1}{2}z(\cos \frac{\pi z}{2})^2 + (3z+1)(\sin \frac{\pi z}{2})^2.$$

They also introduce more complicated entire functions that interpolate the function $\phi(x)$ at odd integer values.

$$F(z) = \frac{4}{\pi^2}(\cos \frac{\pi z}{2})^2 \left(\sum_{n=0}^{\infty} [\alpha_n \frac{1}{(z - (2n+1))^2} - \frac{1}{(2n+1)^2}] \right)$$

in which $\alpha(n) := \phi(2n+1)$. They raise the problem of determining the Fatou and Julia sets of these functions. These functions are constructed so that the positive integer values (resp. positive odd integer values) fall in the Fatou set.

153. Toshio Urata, Katsufumi Suzuki and Hisao Kajita (1997), *Collatz Problem II*, (Japanese), Epsilon (The Bulletin of the Society of Mathematics Education of Aichi Univ. of Education) [Aichi Kyoiku Daigaku Sugaku Kyoiku Gakkai shi] **39** (1997), 121–129.

The authors study the speeded-up Collatz function $\phi(x) = \frac{3x+1}{2^e}$ which takes odd integers to odd integers. They observe that every orbit of $\phi(x)$ contains some integer $n \equiv 1 \pmod{4}$. They then introduce entire functions that interpolate $\phi(x)$ at positive odd integers. They start with

$$F(z) = \frac{4}{\pi^2}(\cos \frac{\pi z}{2})^2 \left(\sum_{n=0}^{\infty} [\alpha_n \frac{1}{(z - (2n+1))^2} - \frac{1}{(2n+1)^2}] \right)$$

in which $\alpha(n) := \phi(2n+1)$, so that $F(2n+1) = \phi(2n+1)$ for $n \geq 0$. They observe $F(z)$ has an attracting fixed point at $z_0 = -0.0327$, a repelling fixed point at $z = 0$, and a superattracting fixed point at $z = 1$. They show that the positive odd integers are in the Fatou set of $F(z)$. They show that the immediate basins of the Fatou set around the positive odd integers of $F(z)$ are disjoint. Computer drawn pictures are included, which include small copies of sets resembling the Mandelbrot set.

The authors also introduce, for each integer $p \geq 2$, the entire functions

$$K_p(z) = \left(\frac{4}{\pi^2}(\cos \frac{\pi z}{2})^2 \right)^p \left(\sum_{n=0}^{\infty} \frac{\alpha_n}{(z - (2n+1))^{2p}} \right),$$

and, for $p \geq 1$, the entire functions.

$$L_p(z) = \left(\frac{4}{\pi^2} \cos^2 \frac{\pi z}{2} \right)^p \frac{\sin \pi z}{\pi} \left(\sum_{n=0}^{\infty} \frac{\alpha_n}{(z - (2n+1))^{2p+1}} \right)$$

These functions also interpolate $\phi(x)$ at odd integers, i.e. $\phi(2n+1) = K_p(2n+1) = L_p(2n+1)$ for $n \geq 0$.

154. Toshio Urata and Hisao Kajita (1998), *Collatz Problem III*, (Japanese), Epsilon (The Bulletin of the Society of Mathematics Education of Aichi Univ. of Education) [Aichi Kyoiku Daigaku Sugaku Kyoiku Gakkai shi] **40** (1998), 57–65.

The authors study the speeded-up Collatz function $\phi(x) = \frac{3x+1}{2^e}$ which takes odd integers to odd integers. They describe the infinite number of preimages of a given x as $\{\frac{4^n x - 1}{3} : n \geq 0\}$ if $x \equiv 1 \pmod{3}$ and $\{\frac{4^n(2x) - 1}{3} : n \geq 0\}$ if $x \equiv 2 \pmod{3}$. They encode the trajectory of this function with a vector (p_1, p_2, p_3, \dots) which keeps track of the powers of 2 divided out at each iteration. More generally they consider the speeded-up $(ax + d)$ -function, with a, d odd. For y_0 an initial value (with $y_0 \neq -\frac{d}{a}$), the iterates are $y_i = \frac{ay_{i-1} + d}{2^{p_i+1}}$. They encode this iteration for n steps by an $n \times n$ matrix

$$\Theta := \begin{bmatrix} 2^{p_1} & 0 & 0 & \cdots & 0 & -a \\ -a & 2^{p_2} & 0 & \cdots & 0 & 0 \\ 0 & -a & 2^{p_3} & \cdots & 0 & 0 \\ & \cdots & & \cdots & & \\ 0 & 0 & 0 & \cdots & -a & 2^{p_n} \end{bmatrix},$$

which acts by

$$\Theta \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \cdots \\ y_n \end{bmatrix} = \begin{bmatrix} d - a(y_n - y_0) \\ d \\ d \\ \cdots \\ d \end{bmatrix}.$$

The authors observe that $\det(\Theta) = 2^{p_1 + \cdots + p_n} - a^n$. A periodic orbit, one with $y_n = y_0$, corresponds to a vector of iterates mapping to the vector with constant entries d . They compute examples for periodic orbits of $5x + 1$ problem, and the $3x + 1$ problem on negative integers. Finally they study orbits with y_0 a rational number with denominator relatively prime to a , and give some examples of periodic orbits.

155. Toshio Urata (1999), *Collatz Problem IV*, (Japanese), Epsilon (The Bulletin of the Society of Mathematics Education of Aichi Univ. of Education) [Aichi Kyoiku Daigaku Sugaku Kyoiku Gakkai shi] **41** (1999), 111–116.

The author studies a 2-adic interpolation of the speeded-up Collatz function $\phi(n)$ defined on odd integers n by dividing out all powers of 2, i.e. for an odd integer n , $\phi(n) = \frac{3n+1}{2^{p(3n+1)}}$, where $p(m) = \text{ord}_2(m)$. Let $\mathbb{Z}_2^* = \{x \in \mathbb{Z}_2 : x \equiv 1 \pmod{2}\}$ denote the 2-adic units. The author sets $OQ := \mathbb{Q} \cap \mathbb{Z}_2^*$, and one has \mathbb{Z}_2^* is the closure \overline{OQ} of $OQ \subset \mathbb{Z}_2$. The author shows that the map ϕ uniquely extends to a continuous function $\phi : \mathbb{Z}_2^* \setminus \{-\frac{1}{3}\} \rightarrow \mathbb{Z}_2^*$. He shows that if $f(x) = 2x + \frac{1}{3}$ then $f(x)$ leaves ϕ invariant, in the sense that $\phi(f(x)) = \phi(x)$ for all $x \in \mathbb{Z}_2^* \setminus \{-\frac{1}{3}\}$. It follows that $f(f(x)) = 4x + 1$ also leaves ϕ invariant.

To each $x \in \mathbb{Z}_2^* \setminus \{-\frac{1}{3}\}$ he associates the sequence of 2-exponents (p_1, p_2, \dots) produced by iterating ϕ . He proves that an element $x \in \mathbb{Z}_2^* \setminus \{-\frac{1}{3}\}$ uniquely determine x ; and that every possible sequence corresponds to some value $x \in \mathbb{Z}_2^* \setminus \{-\frac{1}{3}\}$. He shows that all periodic points of ϕ on \mathbb{Z}_2^* are rational numbers $x = \frac{p}{q} \in OQ$, and that there is a unique such periodic point for any finite sequence (p_1, p_2, \dots, p_m) of positive integers,

representing 2-exponents, having period m . If $C(p_1, p_2, \dots, p_m) = \sum_{j=0}^{m-1} 2^{p_1+\dots+p_j} 3^{m-1-j}$ then this periodic point is

$$x = R(p_1, p_2, \dots, p_m) := \frac{C(p_1, \dots, p_m)}{2^{p_1+\dots+p_m} - 3^m}$$

He shows that an orbit is periodic if and only if its sequence of 2-exponents is periodic.

156. Giovanni Venturini (1982), *Sul Comportamento delle Iterazioni di Alcune Funzioni Numeriche*, Rend. Sci. Math. Institute Lombardo **A 116** (1982), 115–130. (MR 87i:11015; Zbl. 583.10009).

The author studies functions $g(n) = a_r n + b_r$ for $n \equiv r \pmod{p}$ where a_r ($0 \leq r \leq p$) are positive rationals with denominator p . He mainly treats the case that the a_r take two distinct values. If $\tau = (a_0 a_1 \cdots a_{p-1})^{1/p}$ has $\tau < 1$ then for almost all n there is some k with $g^{(k)}(n) < n$, while if $\tau > 1$ then the iterates tend to increase. [The Zentralblatt reviewer says that proofs are incomplete but contain an interesting idea. Rigorous versions of these results have since been established, see Lagarias (1985), Sect. 3.2.]

157. Giovanni Venturini (1989), *On the $3x+1$ Problem*, Adv. Appl. Math **10** (1989), 344–347. (MR 90i:11020).

This paper shows that for any fixed ρ with $0 < \rho < 1$ the set of $m \in \mathbb{Z}^+$ which either have some $T^{(i)}(m) = 1$ or some $T^{(i)}(m) < \rho m$ has density one. This result improves on Dolan, Gilman and Manickam (1987).

158. Giovanni Venturini (1992), *Iterates of Number Theoretic Functions with Periodic Rational Coefficients (Generalization of the $3x+1$ problem)*, Studies in Applied Math. **86** (1992), 185–218. (MR 93b:11102).

This paper studies iteration of maps $g : \mathbb{Z} \rightarrow \mathbb{Z}$ of the form $g(m) = \frac{1}{d}(a_r m + b_r)$ if $m \equiv r \pmod{d}$ for $0 \leq r \leq d-1$, where $d \geq 2$ is an arbitrary integer, and all $a_r, b_r \in \mathbb{Z}$. These maps generalize the $(3x+1)$ -function, and include a wider class of such functions than in Venturini (1989). The author's methods, starting in section 3, are similar in spirit to the Markov chain methods introduced by Leigh (1985), which in turn were motivated by work of Matthews and Watts (1984, 1985).

The author is concerned with classifying g -ergodic sets S of such g which are finite unions of congruence classes. For example, the mapping $g(3m) = 2m$, $g(3m+1) = 4m+3$ and $g(3m+2) = 4m+1$ is a permutation and has $S = \{m : m \equiv 0 \text{ or } 5 \pmod{10}\}$ as a g -ergodic set. He then associates a (generally finite) Markov chain to such a g -ergodic set, whose stationary distribution is used to derive a conjecture for the distribution of iterates $\{g^{(k)}(n_0) : k \geq 1\}$ in these residue classes for a randomly chosen initial value n_0 in S . For the example above the stationary distribution is $p_0 = \frac{1}{3}$ and $p_5 = \frac{2}{3}$. One can obtain also conjectured growth rates $a(g|_S)$ for iterates of a randomly chosen initial value in a g -ergodic set S . For the example above one obtains $a(g|_S) = (\frac{4}{3})^{2/3} (\frac{2}{3}) \cong 1.0583$.

The author classifies maps g into classes $G_v(d)$, for $v = 0, 1, 2, \dots$, with an additional class $G_\infty(d)$. The parameter v measures the extent to which the numerators a_r of iterates of $g(v)$ have common factors with d . The class $G_0(d)$ consists of those maps g having $\gcd(a_0 a_1 \dots a_{d-1}, d) = 1$, which is exactly the *relatively prime case* treated in Matthews and

Watts (1984) and the class $G_1(d)$ are exactly those maps having $\gcd(a_r, d^2) = \gcd(a_r, d)$ for all r , which were the class of maps treated in Matthews and Watts (1985). His Theorem 7 shows that for each finite v both of Leigh's Markov chains for auxiliary modulus $m = d$ are finite for all maps in $G_v(d)$ (strengthening Leigh's Theorem 7). The maps in the exceptional class $G_\infty(d)$ sometimes, but not always, lead to infinite Markov chains. The class $G_\infty(d)$ presumably contains functions constructed by J. H. Conway [Proc. 1972 Number Theory Conf., U. of Colorado, Boulder 1972, 39-42] which have all $b_r = 0$ and which can encode computationally undecidable problems.

The proof of the author's Corollary to Theorem 6 is incomplete and the result is not established: It remains an open problem whether a \mathbb{Z} -permutation having an ergodic set S with $a(g|_S) > 1$ contains any orbit that is infinite. Sections 6 and 7 of the paper contain many interesting examples of Markov chains associated to such functions g ; these examples are worth looking at for motivation before reading the rest of the paper.

159. Giovanni Venturini (1997), *On a Generalization of the $3x+1$ problem*, Adv. Appl. Math. **19** (1997), 295–305. (MR 98j:11013).

This paper considers mappings $T(x) = \frac{t_r x - u_r}{p}$ when $x \equiv r \pmod{p}$, having $t_r \in \mathbb{Z}^+$, and $u_r \equiv r t_r \pmod{p}$. It shows that if $\gcd(t_0 t_1 \dots t_{p-1}, p) = 1$ and $t_0 t_1 \dots t_{p-1} < p^p$ then for any fixed ρ with $0 < \rho < 1$ almost all $m \in \mathbb{Z}$ have an iterate k with $|T^{(k)}(m)| < \rho|m|$. The paper also considers the question: when are such mappings T permutations of \mathbb{Z} ? It proves they are if and only if $\sum_{r=0}^{p-1} \frac{1}{t_r} = 1$ and $T(r) \not\equiv T(s) \pmod{(t_r, t_s)}$ for $0 \leq r < s \leq p-1$. The geometric-harmonic mean inequality implies that $t_0 \dots t_{p-1} > p^p$ for such permutations, except in the trivial case that all $t_i = p$.

160. Carlo Viola (1983), *Un Problema di Aritmetica (A problem of arithmetic)* (Italian), Archimede **35** (1983), 37–39. (MR 85j:11024).

The author states the $3x+1$ problem, and gives a very brief survey of known results on the problem, with pointers to the literature.

161. Stanley Wagon (1985), *The Collatz problem*, Math. Intelligencer **7**, No. 1, (1985), 72–76. (MR 86d:11103, Zbl. 566.10008).

This article studies a random walk imitation of the “average” behavior of the $3x+1$ function, computes its expected value and compares it to data on $3x+1$ iterates.

162. Wang, Shi Tie, *Some researches for transformation over recursive programs*, (Chinese, English summary) J. Xiamen University, Natural Science Ed. [Xiamen da xue xue bao. Zi ran ke xue ban] **27**, No. 1 (1988), 8–12. [MR 89g:68057, Zbl. 0689.68010]

English Abstract: "In this paper the transformations over recursive programs with the fixpoint theory is reported. The termination condition of the duple-recursive programs to compute the "91" function and to prove the $3x+1$ problem is discussed."

Note. The author gives a general iteration scheme for computing certain recursively defined functions. The 91 function $F(x)$ is defined recursively on positive integers by the condition, if $x > 100$ then $F(x) := x - 10$, otherwise $F(x) := F(F(x + 11))$.

163. Blanton C. Wiggin (1988), *Wondrous Numbers – Conjecture about the $3n + 1$ family*, J. Recreational Math. **20**, No. 2 (1988), 52–56.

This paper calls Collatz function iterates “Wondrous Numbers” and attributes this name to D. Hofstadter, *Gödel, Escher, Bach*. He proposes studying iterates of the class of “MU” functions

$$F_D(x) = \begin{cases} \frac{x}{D} & \text{if } x \equiv 0 \pmod{D} , \\ (D+1)x - j & \text{if } x \equiv j \pmod{D}, \ 1 \leq j \leq D-2 , \\ (D+1)x + 1 & \text{if } x \equiv -1 \pmod{D} , \end{cases}$$

F_D is the Collatz function for $D = 2$. Wiggin’s analogue of the $3x + 1$ conjecture for a given $D \geq 2$ is that all iterates of $F_D(n)$ for $n \geq 1$ reach some number smaller than D . Somewhat surprisingly, no D is known for which this is false. It could be shown false for a given D by exhibiting a cycle with all members $> D$; no such cycles exist for $x < 3 \times 10^4$ for $2 \leq D \leq 12$.

164. Günther J. Wirsching (1993), *An Improved Estimate Concerning $3N + 1$ Predecessor Sets*, Acta Arithmetica, **63** (1993), 205–210. (MR 94e:11018).

This paper shows that, for all $a \not\equiv 0 \pmod{3}$, the set $\theta_a(x) := \{n \leq x : \text{some } T^{(k)}(x) = a\}$ has cardinality at least $x^{.48}$, for sufficiently large x . This is achieved by exploiting the inequalities of Krasikov (1989).

165. Günther J. Wirsching (1994), *A Markov chain underlying the backward Syracuse algorithm*, Rev. Roumaine Math. Pures Appl. **39** (1994), no. 9, 915–926. (MR 96d:11027).

The author constructs from the inverse iterates of the $3x + 1$ function (‘backward Syracuse algorithm’) a Markov chain defined on the state space $[0, 1] \times \mathbb{Z}_3^\times$, in which \mathbb{Z}_3^\times is the set of invertible 3-adic integers. He lets $g_n(k, a)$ count the number of “small sequence” preimages of an element $a \in \mathbb{Z}_3^\times$ at depth $n + k$, which has n odd iterates among its preimages, and with symbol sequence $0^{\alpha_0} 10^{\alpha_1} \dots 10^{\alpha_{n-1}} 1$, with $\sum_{i=0}^n \alpha_i = k$ satisfying the “small sequence” condition $0 \leq \alpha_j < 2 \cdot 3^{j-1}$ for $0 \leq j \leq n - 1$. These quantities satisfy a functional equation

$$g_n(k, a) = \frac{1}{2 \cdot 3^{n-1}} \sum_{j=0}^{2 \cdot 3^{n-1}} g_{n-1} \left(k - j, \frac{2^{j+1}a - 1}{3} \right) .$$

He considers the “renormalized” quantities

$$\hat{g} \left(\frac{k}{n}, a \right) := \frac{1}{\Gamma_n} g_n(k, a) ,$$

with

$$\Gamma_n = 2^{1-n} 3^{-\frac{1}{2}(n-1)(n-2)} (3^n - n) ,$$

He obtains, in a weak limiting sense, a Markov chain whose probability density of being at (x, a) is a limiting average of $\hat{g}(x, a)$ in a neighborhood of (x, a) as the neighborhood

shrinks to (x, α) . One step of the backward Syracuse algorithm induces (in some limiting average sense) a limiting Markovian transition measure, which has a density taking the form of a product measure $\frac{3}{2}\chi_{[\frac{x}{3}, \frac{x+2}{3}]} \otimes \phi$, in which $\chi_{[\frac{x}{3}, \frac{x+2}{3}]}$ is the characteristic function of the interval $[\frac{x}{3}, \frac{x+2}{3}]$ and ϕ is a nonnegative integrable function on \mathbb{Z}_3^\times (which may take the value $+\infty$).

The results of this paper are included in Chapter IV of Wirsching (1996b).

166. Günther J. Wirsching (1996), *On the Combinatorial Structure of $3N+1$ Predecessor Sets*, Discrete Math. **148** (1996), 265–286. (MR 97b:11029).

This paper studies the set $P(a)$ of inverse images of an integer a under the $3x+1$ function. Encode iterates $T^{(k)}(n) = a$ by a set of nonnegative integers $(\alpha_0, \alpha_1, \dots, \alpha_\mu)$ such that the 0-1 vector \mathbf{v} encoding $\{T^{(j)}(n) \pmod{2}: 0 \leq j \leq k-1\}$ has $\mathbf{v} = 0^{\alpha_0} 1 0^{\alpha_1} \dots 1 0^{\alpha_\mu}$. Wirsching studies iterates corresponding to “small sequences,” which are ones with $0 \leq \alpha_i < 2 \cdot 3^{i-1}$. He lets $G_\mu(a)$ denote the set of “small sequence” preimages n of a having a fixed number μ of iterates $T^{(j)}(n) \equiv 1 \pmod{2}$, and shows that $|G_\mu(a)| = 2^{\mu-1} 3^{1/2(\mu-1 \dots \mu-2)}$. He lets $g_a(k, \mu)$ denote the number of such sequences with $k = \alpha_0 + \alpha_1 + \dots + \alpha_\mu$, and introduces related combinatorial quantities $\psi(k, \mu)$ satisfying $\sum_{0 \leq l \leq k} \psi(l, \mu-1) \geq g_a(k, \mu)$. The quantities $\psi(k, \mu)$ can be asymptotically estimated, and have a (normalized) limiting distribution

$$\psi(x) = \lim_{\mu \rightarrow \infty} \frac{3^\mu - \mu}{2^\mu 3^{1/2\mu(\mu-1)}} \psi([3^\mu - \mu]x, \mu).$$

$\psi(x)$ is supported on $[0, 1]$ and is a C^∞ -function. He suggests a heuristic argument to estimate the number of “small sequence” preimages of a smaller than $2^n a$, in terms of a double integral involving the function $\psi(x)$.

The results of this paper are included in Chapter IV of Wirsching (1998).

167. Günther J. Wirsching (1997), *$3n+1$ Predecessor Densities and Uniform Distribution in \mathbb{Z}_3^** , Proc. Conference on Elementary and Analytic Number Theory (in honor of E. Hlawka), Vienna, July 18-20, 1996, (G. Nowak and H. Schoissengeier, Eds.), 1996, pp. 230-240. (Zbl. 883.11010).

This paper formulates a kind of equidistribution hypothesis on 3-adic integers under backwards iteration by the $3x+1$ mapping, which, if true, would imply that the set of integers less than x which iterate under T to a fixed integer $a \not\equiv 0 \pmod{3}$ has size at least $x^{1-\epsilon}$ as $x \rightarrow \infty$, for any fixed $\epsilon > 0$.

168. Günther J. Wirsching (1998a), *The Dynamical System Generated by the $3n+1$ Function*, Lecture Notes in Math. No. 1681, Springer-Verlag: Berlin 1998. (MR 99g:11027).

This volume is a revised version of the author’s Habilitationsschrift (Katholische Universität Eichstätt 1996). It studies the problem of showing that for each positive integer $a \not\equiv 0 \pmod{3}$ a positive proportion of integers less than x iterate to a , as $x \rightarrow \infty$. It develops an interesting 3-adic approach to this problem.

Chapter I contains a history of work on the $3x+1$ problem, and a summary of known results.

Chapter II studies the graph of iterates of T on the positive integers, (“Collatz graph”)

and, in particular studies the graph $\Pi^a(\Gamma_T)$ connecting all the inverse iterates $P(a)$ of a given positive integer a . This graph is a tree for any noncyclic value a . Wirsching uses a special encoding of the symbolic dynamics of paths in such trees, which enumerates symbol sequences by keeping track of the successive blocks of 0's. He then characterizes which graphs $\Pi^a(\Gamma_T)$ contain a given symbol sequence reaching the root node a . He derives “counting functions” for such sequences, and uses them to obtain a formula giving a lower bound for the function

$$P_T^n(a) := \{m : 2^n a \leq m < 2^{n+1} a \text{ and some iterate } T^{(j)}(m) = a\} ,$$

which states that

$$|P_T^n(a)| := \{m : 2^n(a) := \sum_{\ell=0}^{\infty} e_{\ell}(n + \lfloor \log_2 \left(\frac{3}{2}\right) \ell \rfloor , a) , \quad (1)$$

where $e_{\ell}(k, a)$ is a “counting function.” (Theorem II.4.9). The author’s hope is to use (1) to prove that

$$|P_T^n(a)| > c(a)2^n , \quad n = 1, 2, \dots . \quad (2)$$

for some constant $c(a) > 0$.

Chapter III studies the counting functions appearing in the lower bound above, in the hope of proving (2). Wirsching observes that the “counting functions” $e_{\ell}(k, a)$, which are ostensibly defined for positive integer variables k, ℓ, a , actually are well-defined when the variable a is a 3-adic integer. He makes use of the fact that one can define a ‘Collatz graph’ $\Pi^a(\Gamma_T)$ in which a is a 3-adic integer, by taking a suitable limiting process. Now the right side of (1) makes sense for all 3-adic integers, and he proves that actually $s_n(a) = +\infty$ on a dense set of 3-adic integers a . (This is impossible for any integer a because $|P_T^n(a)| \leq 2^n a$.) He then proves that $s_n(a)$ is a nonnegative integrable function of a 3-adic variable, and he proves that its expected value \bar{s}_n is explicitly expressible using binomial coefficients. Standard methods of asymptotic analysis are used to estimate \bar{s}_n , and to show that

$$\liminf_{n \rightarrow \infty} \frac{\bar{s}_n}{2^n} > 0 , \quad (3)$$

which is Theorem III.5.2. In view of (1), this says that (2) ought to hold in some “average” sense. He then proposes that (2) holds due to the:

Heuristic Principle. *As $n \rightarrow \infty$, $s_n(a)$ becomes relatively close to \bar{s}_n , in the weak sense that there is an absolute constant $c_1 > 0$ such that*

$$s_n(a) > c_1 \bar{s}_n \text{ for all } n > n_0(a) . \quad (4)$$

One expects this to be true for all positive integers a . A very interesting idea here is that it might conceivably be true for all 3-adic integers a . If so, there is some chance to rigorously prove it.

Chapter IV studies this heuristic principle further by expressing the counting function $e_{\ell}(k, a)$ in terms of simpler counting functions $g_{\ell}(k, a)$ via a recursion

$$e_{\ell}(k, a) = \sum_{j=0}^k p_{\ell}(k-j) g_{\ell}(j, a) ,$$

in which $p_\ell(m)$ counts partitions of m into parts of a special form. Wirsching proves that properly scaled versions of the functions g_ℓ “converge” (in a rather weak sense) to a limit function which is independent of a , namely

$$g_\ell(k, a) \approx 2^{\ell-1} 3^{\frac{1}{2}(\ell^2-5\ell+2)} \psi\left(\frac{k}{3^\ell}\right), \quad (5)$$

for “most” values of k and a . (The notion of convergence involves integration against test functions.) The limit function $\psi : [0, 1] \rightarrow \mathbb{R}$ satisfies the functional-differential equation

$$\psi'(t) = \frac{9}{2}(\psi(3t) - \psi(3t-2)). \quad (6)$$

He observes that ψ has the property of being C^∞ and yet is piecewise polynomial, with infinitely many pieces, off a set of measure zero. There is a fairly well developed theory for related functional-differential equations, cf. V. A. . Rvachev, Russian Math. Surveys **45** No. 1 (1990), 87–120. Finally, Wirsching observes that if the sense of convergence in (5) can be strengthened, then the heuristic principle can be deduced, and the desired bound (2) would follow.

169. Günther J. Wirsching (1998b), *Balls in Constrained Urns and Cantor-Like Sets*, Z. für Analysis u. Anwendungen **17** (1998), 979-996. (MR 2000b:05007).

This paper studies solutions to an integral equation that arises in the author’s analysis of the $3x+1$ problem (Wirsching (1998a)). Berg and Krüppel (J. Anal. Appl. **17** (1998), 159–181) showed that the integral equation

$$\phi(x) = \frac{q}{q-1} \int_{qx-q+1}^{qx} \phi(y) dy$$

subject to ϕ being supported in the interval $[0, 1]$ and having $\int_0^1 \phi(y) dy = 1$ has a unique solution whenever $q > 1$. The function $\phi(y)$ is a C^∞ -function. In this paper Wirsching shows that a certain iterative procedure converges to this solution when $q > \frac{3}{2}$. He also shows that for $q > 2$ the function $\phi(y)$ is piecewise polynomial off a Cantor-like set of measure zero. The case of the $3x+1$ problem corresponds to the choice $q = 3$.

170. Wu, Jia Bang (1992), *A tendency of the proportion of consecutive numbers of the same height in the Collatz problem* (Chinese), J. Huazhong (Central China) Univ. Sci. Technol. [Hua zhong gong xue yuan] **20**, No. 5, (1992), 171–174. (MR 94b:11024, Zbl. 766.11013).

English Abstract: ”The density distribution and length of consecutive numbers of the same height in the Collatz problem are studied. The number of integers, K , which belong to n -tuples ($n \geq 2$) in interval $[1, 2^N)$ ($N = 1, 2, \dots, 24$) is accurately calculated. It is found that the density $d(2^N)$ ($= K/2^N - 1$) of K in $[1, 2^N)$ is increased with N . This is a correction of Garner’s inferences and prejudgments. The longest tuple in $[1, 2^{30})$, which is the 176-tuple with initial number 722, 067, 240 has been found. In addition, two conjectures are proposed.”

Note: The author refers to Garner (1985).

171. Wu, Jia Bang (1993), *On the consecutive positive integers of the same height in the Collatz problem* (Chinese), Mathematica Applicata, suppl. **6** [Ying yung shu hsüeh] (1993),

150–153. (MR 1 277 568)

English Abstract: "In this paper, we prove that, if a n -tuple (n consecutive positive integers of the same height) is found, an infinite number of n -tuples can be found such that each of n numbers in the same tuple has the same height. With the help of a computer, the author has checked that all positive integers up to 1.5×10^8 , verified the $3x + 1$ conjecture and found a 120-tuple, which is the longest tuple among all checked numbers. Thus there exists an infinite number of 120-tuples."

Note. A table of record holders over intervals of length 10^7 is included.

172. Wu, Jia Bang (1995), *The monotonicity of pairs of coalescence numbers in the Collatz problem.* (Chinese), J. Huazhong Univ. Sci. Tech. [Hua zhong gong xue yuan] **23** (1995), suppl. II, 170–172. (MR 1 403 509)

English Abstract: "A 310-tuple with an initial number 6,622,073,000 has been found. The concepts such as the pairs of coalescence numbers, conditional pairs of coalescence numbers and unconditional pairs of coalescence numbers are suggested. It is proved that, in interval $[1, 2^N]$, the density of conditional pairs of coalescence numbers

$$\bar{d}(2^N) = \frac{1}{2^N} \# \{n < 2^N : n \text{ and } n+1 \text{ are } k(\leq N) \text{ times the pair of coalescence numbers}\}$$

is increased with N . A conjecture that, in interval $[1, 2^N]$, the density of

$$\bar{d}_0(2^N) = \frac{1}{2^N} \# \{n < 2^N : n \text{ and } n+1 \text{ pair of coalescence number}\}$$

is also increased with N , is proposed."

173. Masaji Yamada (1980), *A convergence proof about an integral sequence*, Fibonacci Quarterly **18** (1980), 231–242. (MR 82d:10026)

This paper claims a proof of the $3x + 1$ Conjecture. However the proof is faulty, with specific mistakes pointed out in Math. Reviews. In particular, Lemma 7 (iii) and Lemma 8 are false as stated.

174. Yang, Zhao Hua (1998), *An equivalent set for the $3x + 1$ conjecture* (Chinese), J. South China Normal Univ. Natur. Sci. Ed. [Hua nan shi fan da xue xue bao. Zi ran ke xue ban] 1998, no. 2, 66–68. (MR 2001f:11040).

The paper shows that the $3x + 1$ Conjecture is true if, for any fixed $k \geq 1$, it is true for all positive integers in the arithmetic progression $n \equiv 3 + \frac{10}{3}(4^k - 1) \pmod{2^{2k+2}}$. Note that Korec and Znam (1987) gave analogous conditions for the $3x + 1$ Conjecture being true, using a residue class $\pmod{p^k}$ where p is an odd prime for which 2 is a primitive root.

175. Yang, Zhi and Zhang, Zhongfu (1988), *Kakutani conjecture and graph theory representation of the black hole problem*, Nature Magazine [Zi ran za zhi (Shanghai)] **11** (1988), No. 6, 453–456.

This paper considers the Collatz function (denoted $J(n)$) viewing iteration of the function as a directed graph, with edges from $n \rightarrow J(n)$. The set N_J of Kakutani numbers is the set of all numbers that iterate to 1; these form a connected graph. It phrases the $3x + 1$ conjecture as asserting that all positive integers are “Kakutani numbers.”

It also discusses the “black hole” problem, which concerns iteration on n -digit integers (in base 10), taking $K(n) = n^+ - n^-$ where n^+ is the n -digit integer obtained from n , arranging digits in decreasing order, and n^- arranging digits in increasing order. This iteration can be arranged in a directed graph with edges from n to $T(n)$. It is known that for 3-digit numbers, $T(\cdot)$ iterates to the fixed point $n = 495$, and for 4-digit numbers it iterates to the fixed point $n = 6174$ (the “Kaprekar constant”). For more than 4-digits, iterations may enter cycles of period exceeding one, called here “black holes”. For $n = 6$ there is a cycle of length 7 with $n = 840852$, and for $n = 9$ there is a cycle of length 14 starting with $n = 864197532$. The paper is descriptive, with examples but no proofs.

Note. The “black hole” problem has a long history. The starting point was: D. R. Kaprekar, *An interesting property of the number 6174*, *Scriba Math.* **21** (1955) 244-245. Relevant papers include H. Hasse and G. D. Prichett, *The determination of all four-digit Kaprekar constants*, *J. reine Angew. Math.* **299/300** (1978), 113-124; G. Prichett, A. Ludington and J. F. Lapenta, *The determination of all decadic Kaprekar constants*, *Fibonacci Quarterly* **19** (1981), 45-52. The latter paper proves that “black holes” exist for every $n \geq 5$.

176. Zhang, Cheng Yu (1990), *A generalization of Kakutani's conjecture*. (Chinese) *Nature Magazine* [Zi ran za zhi (Shanghai)] , **13** No. 5, (1990), 267-269.

Let $p_1 = 2, p_2 = 3, \dots$ list the primes in increasing order. This paper suggests the analysis of the mappings $T_{d,k}(x)$, indexed by $d \geq 1$ and $k \geq 0$, with

$$T_{d,k}(x) = \begin{cases} \frac{x}{2} & \text{if } 2|x \\ \frac{x}{3} & \text{if } 3|x \text{ and } 2 \nmid x \\ \dots & \\ \frac{x}{p_{d-1}} & \text{if } p_d|x \text{ and } p_j \nmid x, \ 1 \leq j \leq d-1, \\ p_{d+1}x + (p_{d-1})^k & \text{otherwise.} \end{cases}$$

This function for $(d, k) = (1, 0)$ is the Collatz function. The author actually does not give a tie-breaking rule for defining the function when several p_j divide x . However, this is not important in studying the conjecture below.

The author formulates the *Kakutani* (d, k) -Conjecture, which states that the function $T_{d,k}(x)$ iterated on the positive integers always reaches the integer $(p_{d+1})^k$. Note that $(p_{d+1})^k$ always belongs to a cycle of the function $T_{d,k}(x)$. This conjecture for $(1, k)$ was presented earlier by the author, where it was shown equivalent to the $3x + 1$ Conjecture.

The author shows this conjecture is false for $d = 4, 5, 6, 8$ because there is a nontrivial cycle. The cycles he finds, for given p_{d+1} and starting value n are: $p_5 = 11$ ($n = 17$), $p_6 = 13$ ($n = 19$), $p_7 = 17$ ($n = 243$), $p_9 = 29$ ($n = 179$).

The author then modifies the Kakutani (d, k) -Conjecture in these cases to say that every orbit on the positive integers enters one of the known cycles. For example he modifies Conjecture $(4, k)$ to say that all orbits reach the value $(11)^k$ or else enter the cycle containing $17 \cdot (11)^k$. He proves that the Conjecture (d, k) is equivalent to the modified Conjecture $(d, 0)$ for $d = 1, 2, 3, 4$.

177. Zhang, Zhongfu and Yang, Zhi (1988), [unknown title] Nature Magazine [Zi ran za zhi (Shanghai)] , **11** (1988), 453.

[I have not seen this paper.]

178. Zhang, Zhongfu and Yang, Shiming (1998), *Problems on mapping sequences* (Chinese), Mathmedia [Shu xue chuan bo ji kan] **22** (1998), No. 2, 76–88.

This paper presents basic results on a number of iteration problems on integers, represented as directed graphs, and lists twelve open problems. This includes a discussion of the $3x + 1$ problem. It also discusses iteration of the map $V(n)$ which equals the difference of numbers given by the decimal digits of n rearranged in decreasing, resp. increasing order. On four digit numbers $V(n)$ iterates to a fixed point 6174, the Kaprekar constant.

179. Zhou, Chuan Zhong (1995), *Some discussion on the $3x + 1$ problem* (Chinese), J. South China Normal Univ. Natur. Sci. Ed. [Hua nan shi fan da xue xue bao. Zi ran ke xue ban] **1995**, No. 3, 103–105. (MR 97h:11021).

Let H be the set of positive integers that eventually reach 1 under iteration by the Collatz function $C(n)$; the $3x + 1$ conjecture states that H consists of all positive integers. This paper extends a result of B. Y. Hong [J. of Hubei Normal University. Natural Science Edition (Hubei shi fan xue yuan xue bao. Zi ran ke xue) 1986, no. 1, 1–5.] by showing that if the $3x + 1$ Conjecture is not true the minimal positive $n \notin H$ that does not iterate to 1 satisfies $n \equiv 7, 15, 27, 31, 39, 63, 79$ or $91 \pmod{96}$. It also proves that if $2^{2m+1} - 1 \in H$ then $2^{2m+2} - 1 \in H$. Finally the paper observes that the numerical computation of $C(n)$ could be simplified if it were performed in base 3.

180. Zhou, Chuan Zhong (1997), *Some recurrence relations connected with the $3x + 1$ problem* (Chinese), J. South China Normal Univ. Natur. Sci. Ed. [Hua nan shi fan da xue xue bao. Zi ran ke xue ban] **1997**, no. 4, 7–8.

English Abstract: “Let \mathbb{N} denote the set of natural numbers, J denote the $3x + 1$ operator, and $H = \{n \in \mathbb{N} : \text{there is } k \in \mathbb{N} \text{ so that } J^k(n) = 1\}$. The conjecture $H = \mathbb{N}$ is the so-called $3x + 1$ problem. In this paper, some recurrence relations on this problems are given.”

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